Systems of Particles

Linear Momentum & Impulse

Unit Outline

I. **Center of Mass**
   discrete, continuous

II. Motion of a System of Particles

III. Conservation of Momentum
   frame of reference

IV. Impulse

V. Variable Mass

	The student will be able to:	HW:
1	Determine the center of mass for a set of objects or particles and/or a continuous distribution of mass.	1 – 7
2	Apply ’s 2nd Law to a system of particles and solve related problems either with the presence or absence of external forces.	8 – 12
3	State and apply the Law of Conservation of Momentum and solve related problems.	13 – 23
4	Define and apply elasticity and solve related problems.	24 – 30
5	Define and apply the concept of impulse and solve problems that relate momentum, force, and impulse.	31 – 38
6	Solve problems involving variable mass such as that of a rocket.	39 – 40

Center of Mass

- Any collection of particles has a unique position called the **center of mass**.
- This position has several unique properties, but it may be thought of as a "balancing point" or an "average position" of all the masses in a particular set of particles.
- In any convenient frame of reference the center of mass is defined by the following:

Position of the Center of Mass

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \ldots}{m_1 + m_2 + m_3 + \ldots}$$

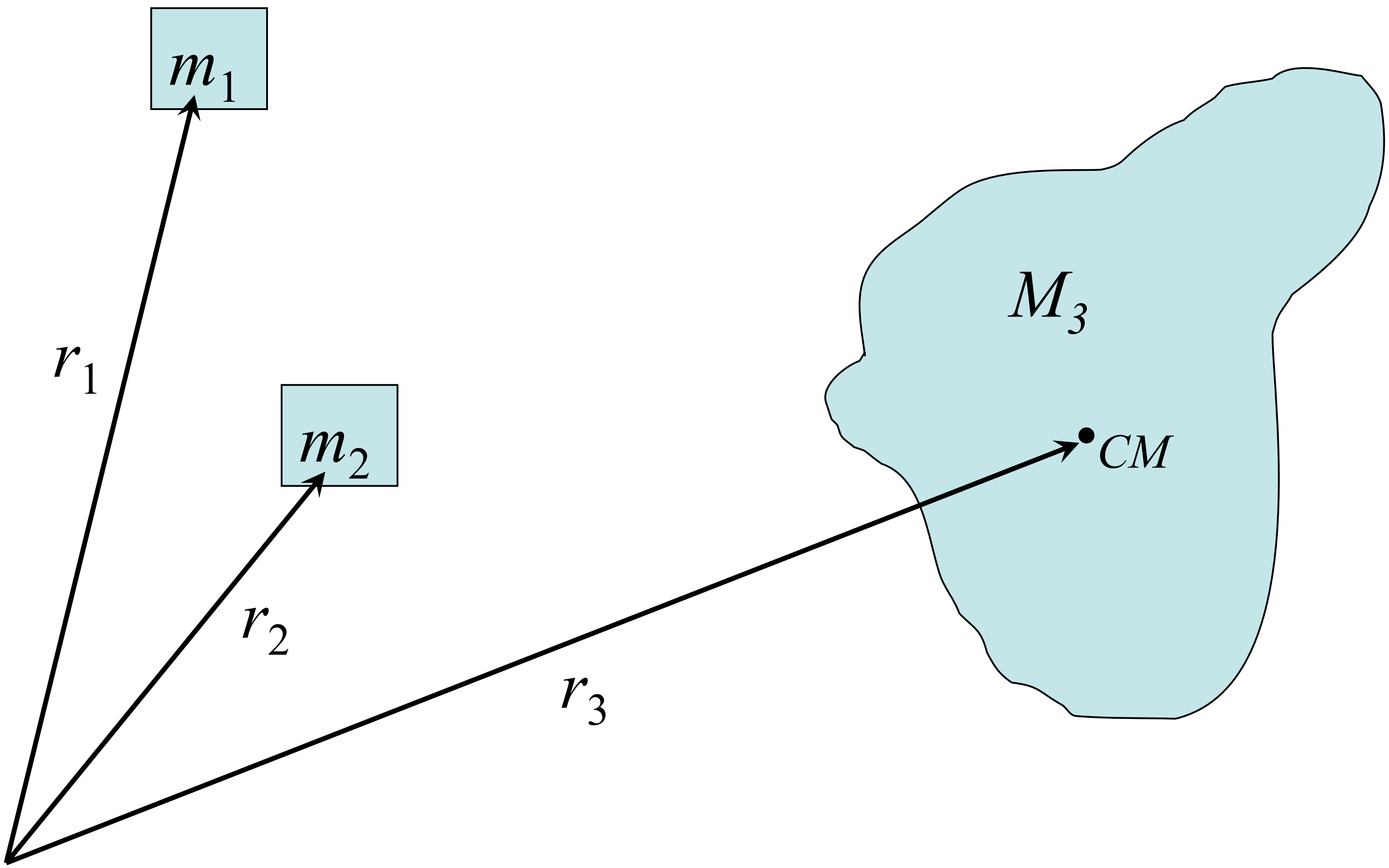
$$\vec{r}_{cm} = \frac{\sum m_i\vec{r}_i}{\sum m_i}$$

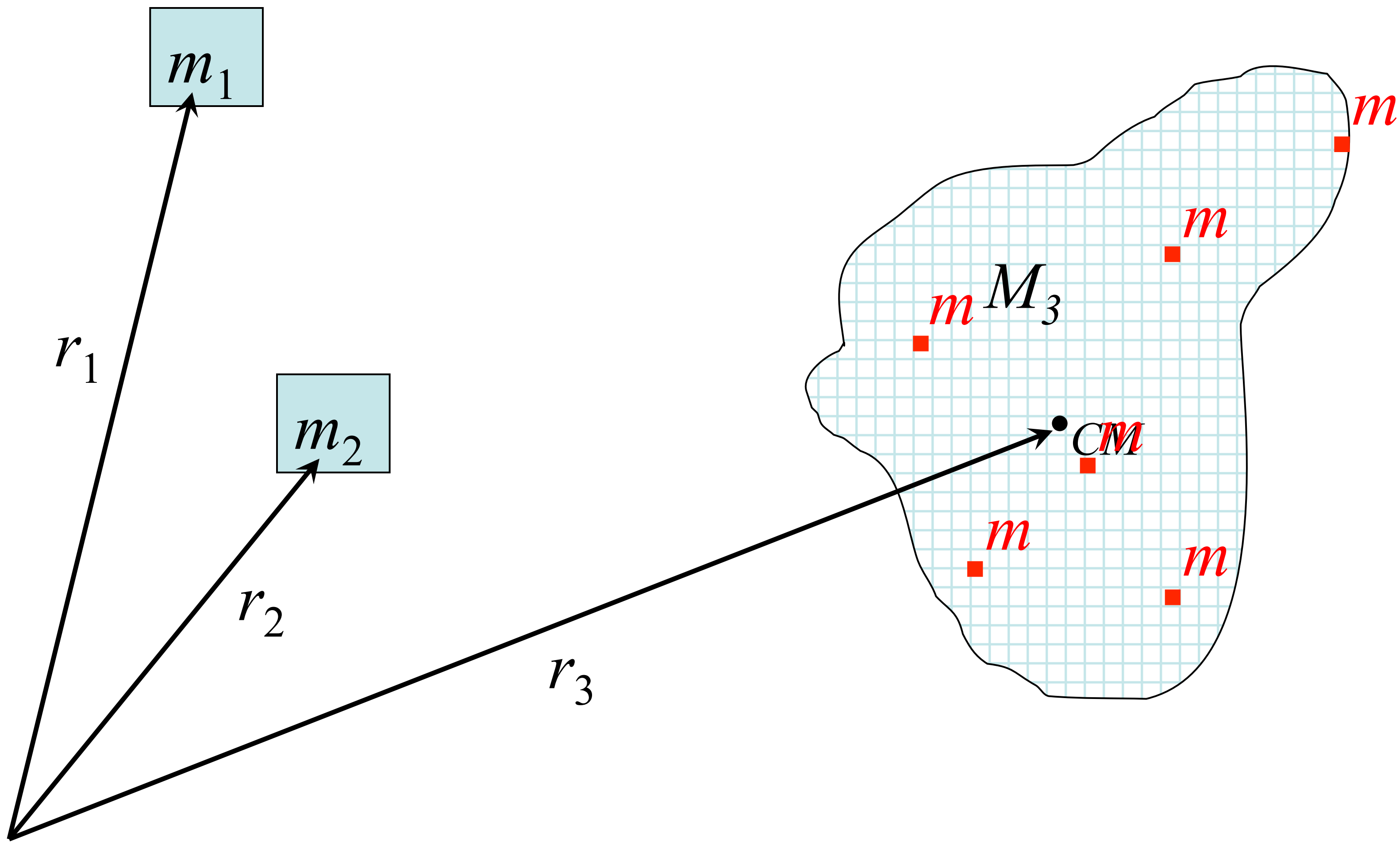
where m = mass, r = position

Using a CM to find a CM

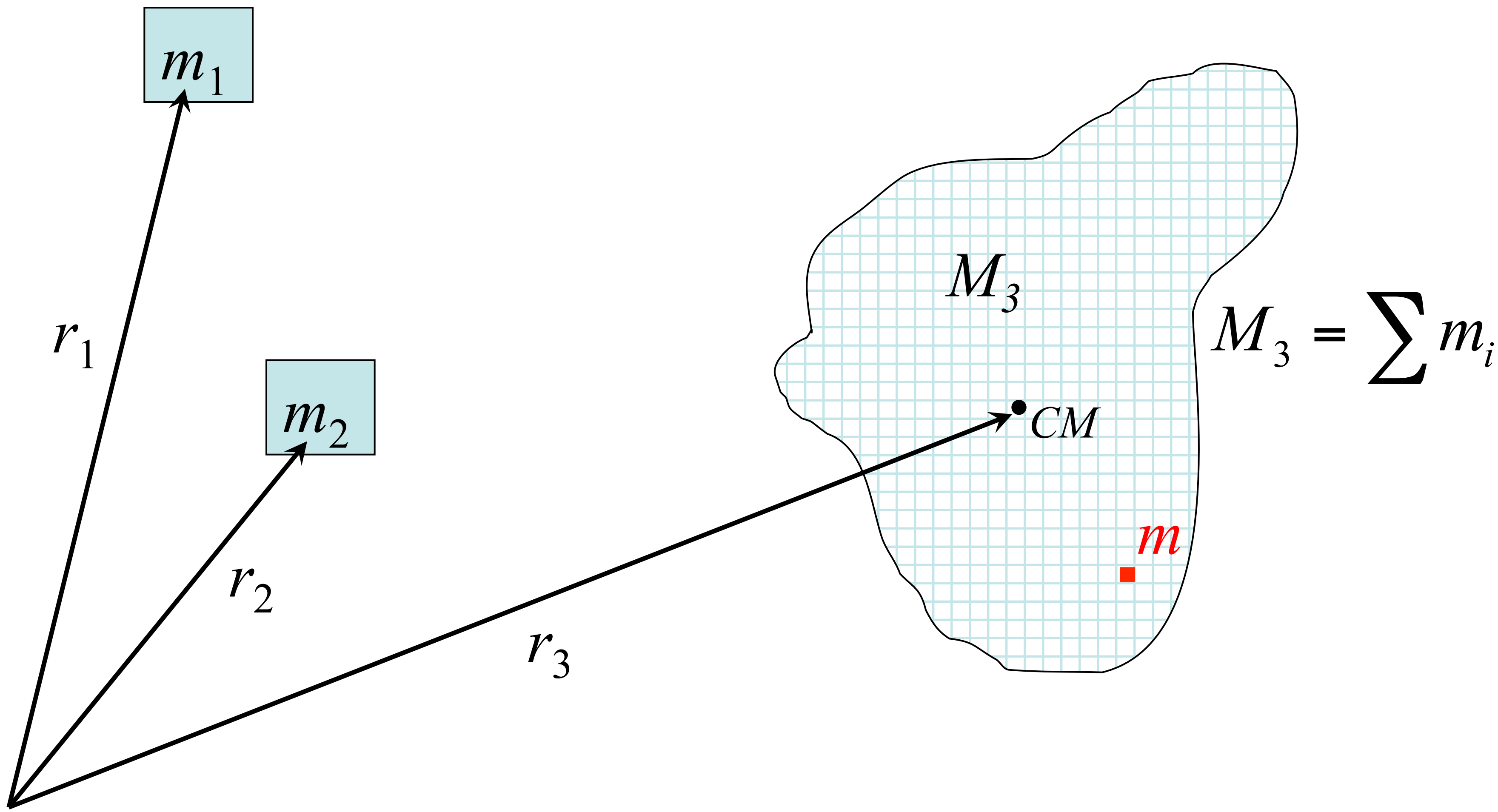
- If the center of mass is known for a collection of masses, it may be used to find a new center of mass for a larger set of masses that contains this collection.
- All of the mass of a set of particles may be considered to reside at the CM of that set for the purpose of finding another CM.

$CM = ?$





M_3 is made of many small masses, m.



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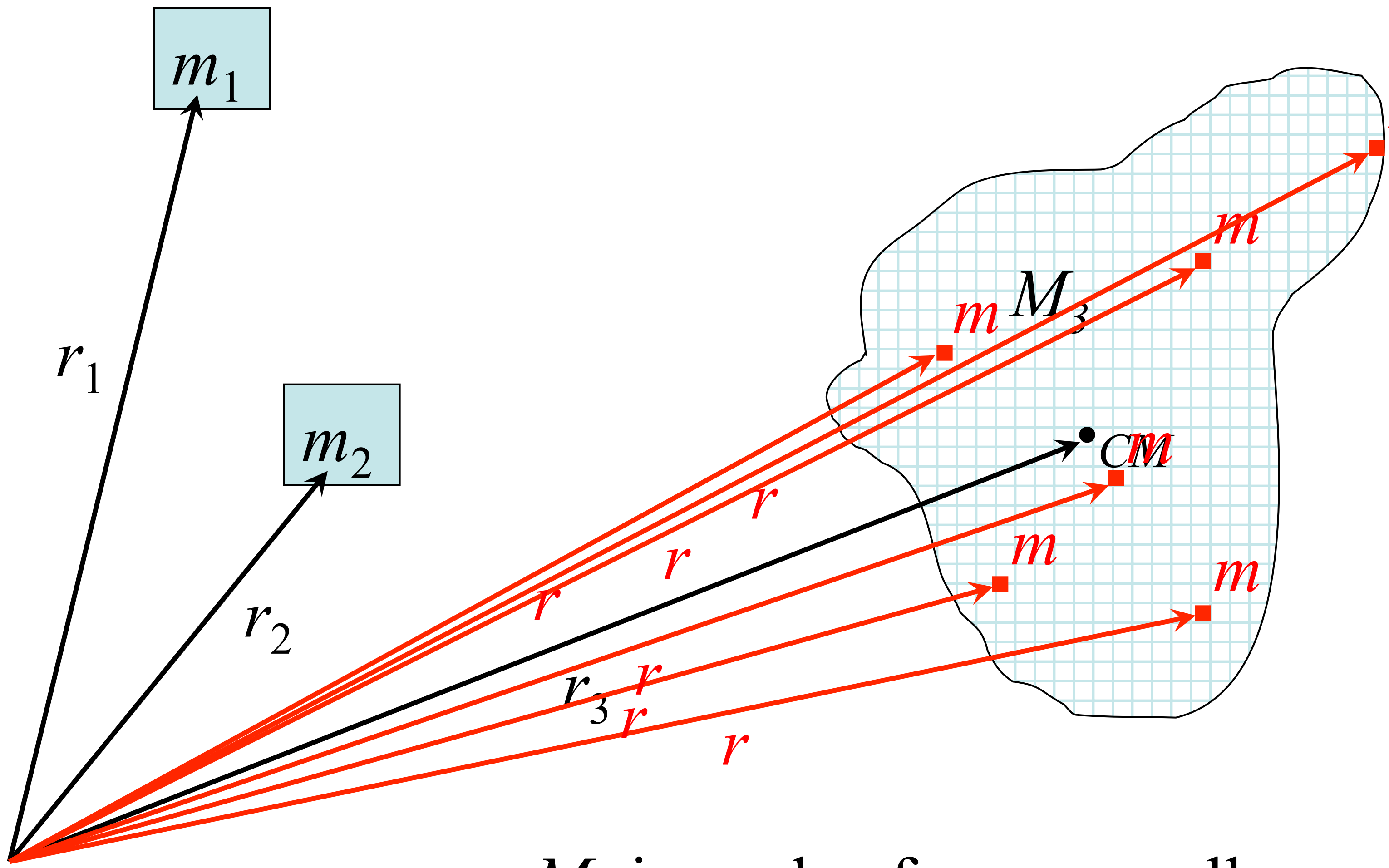
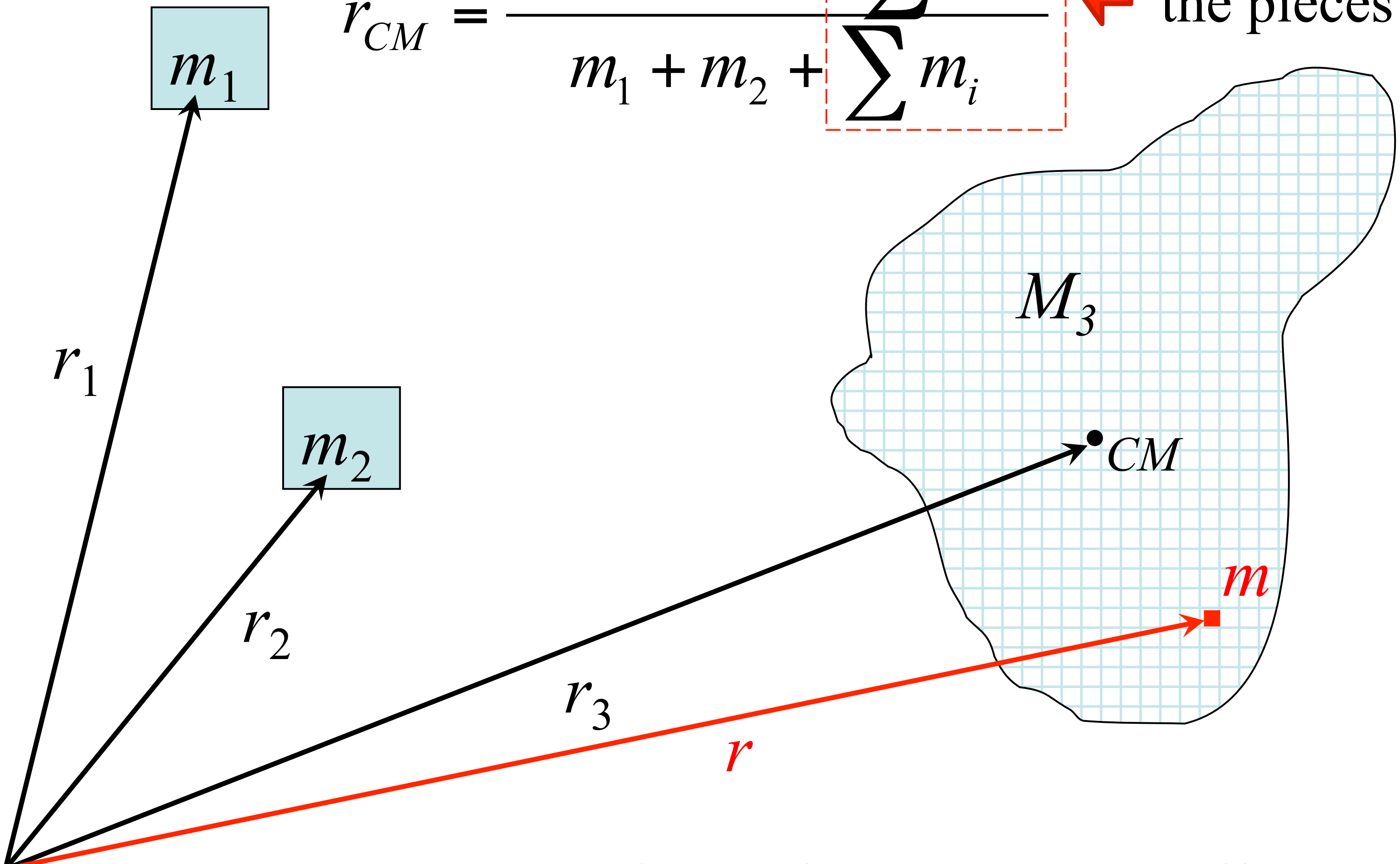


M_3 is made of many small masses, *m*, each with certain position, *r*.

The *CM* of the entire assemblage would incorporate all the pieces of M_3:

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \sum m_i\vec{r}_i}{m_1 + m_2 + \sum m_i}$$

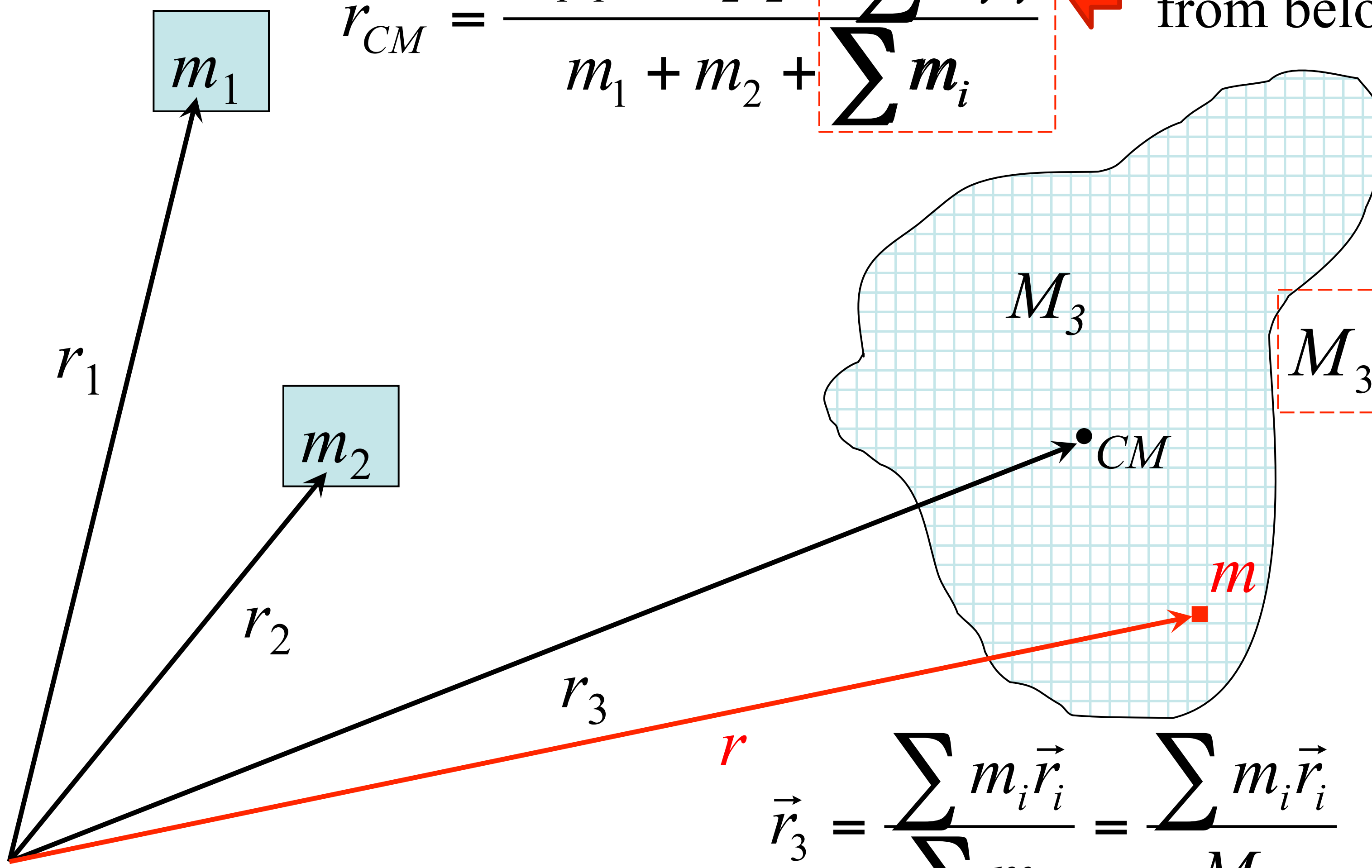
These terms are all the pieces of M_3.



M_3 is made of many small masses, m, each with certain position, r.

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \sum m_i\vec{r}_i}{m_1 + m_2 + \sum m_i}$$

Substitute from below…



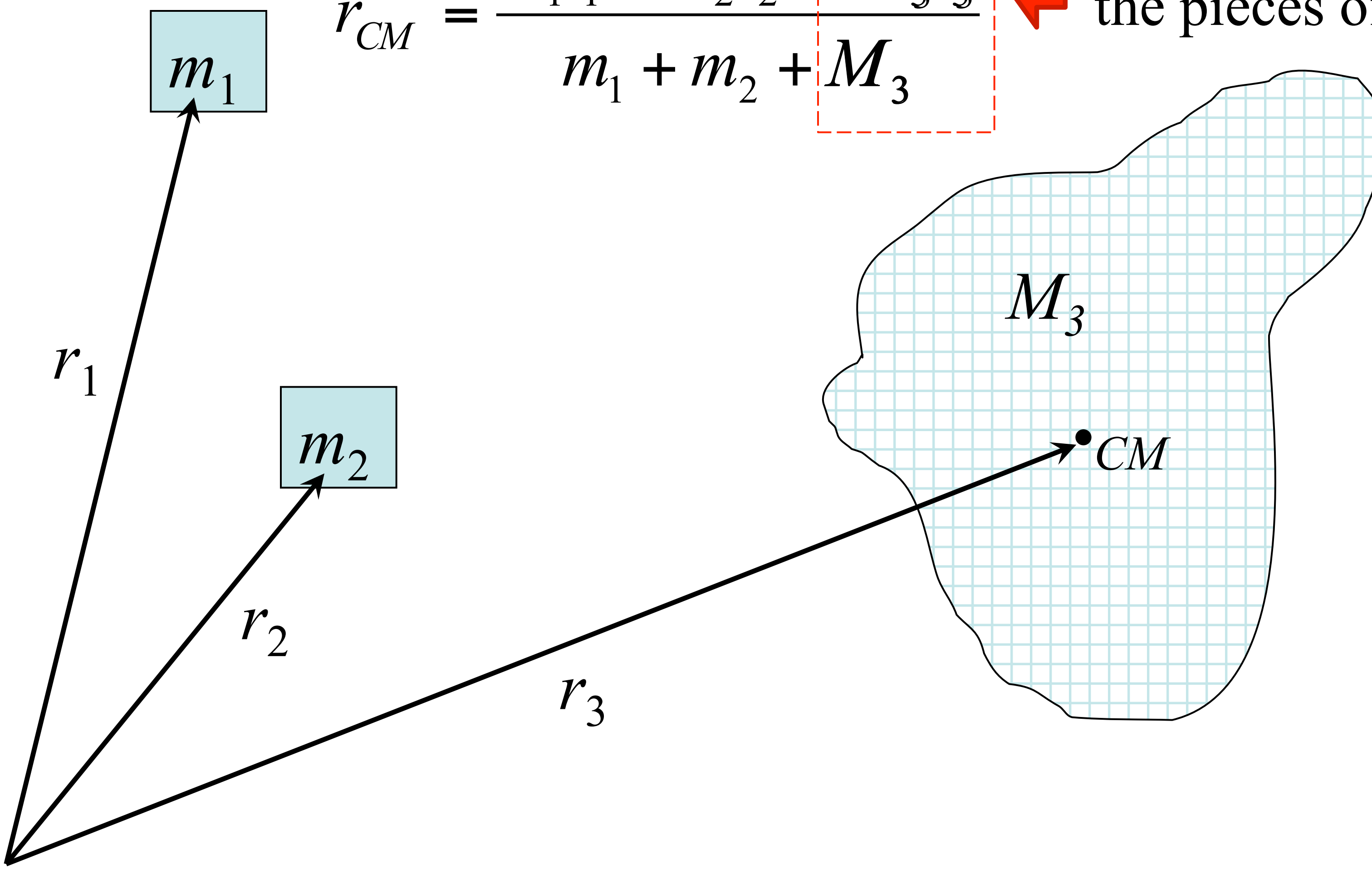
$$M_3 = \sum m_i$$

$$\vec{r}_3 = \frac{\sum m_i\vec{r}_i}{\sum m_i} = \frac{\sum m_i\vec{r}_i}{M_3}$$

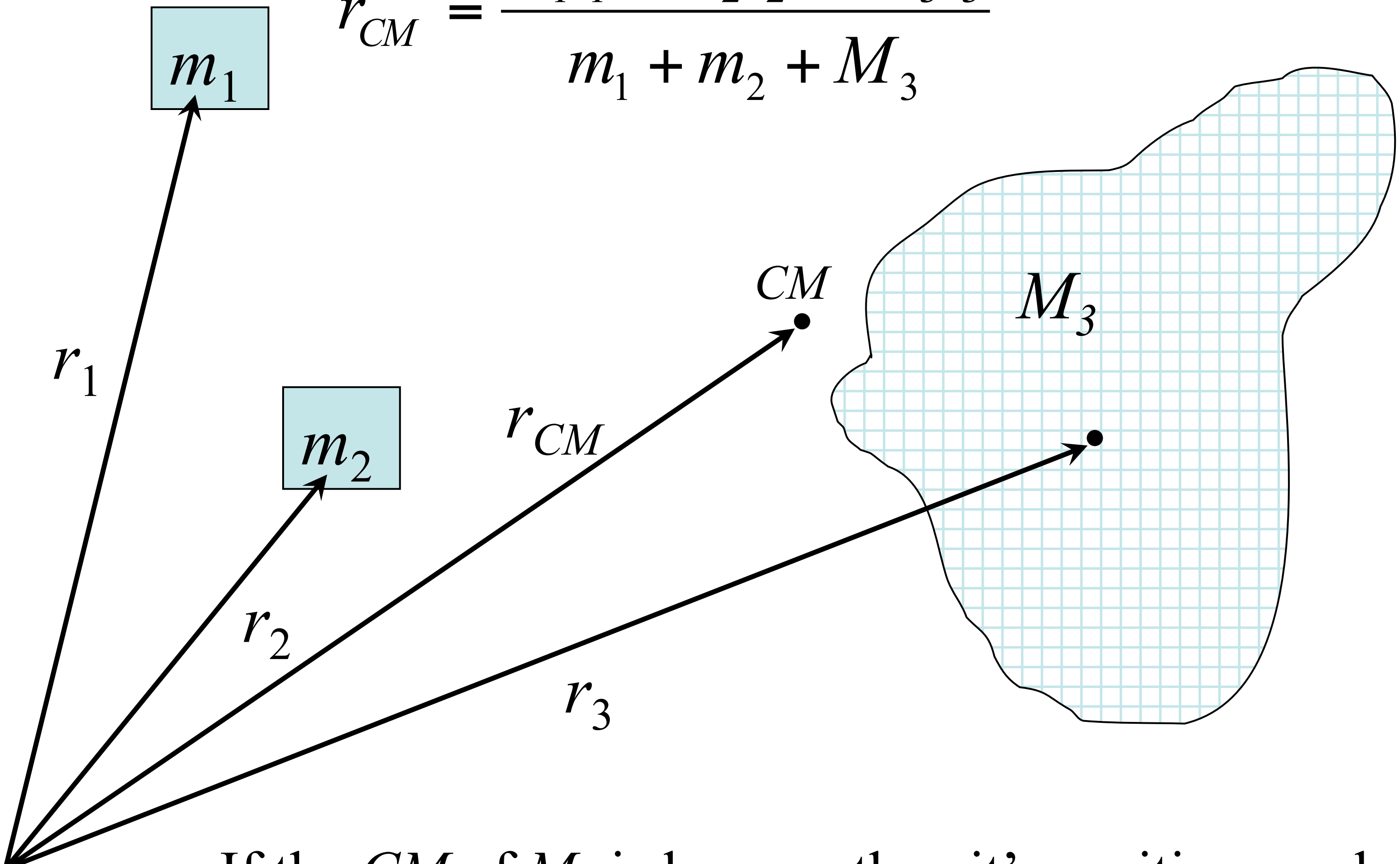
$$M_3\vec{r}_3 = \sum m_i\vec{r}_i$$

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \boldsymbol{M_3\vec{r}_3}}{m_1 + m_2 + \boldsymbol{M_3}}$$

Accounts for all the pieces of M_3.

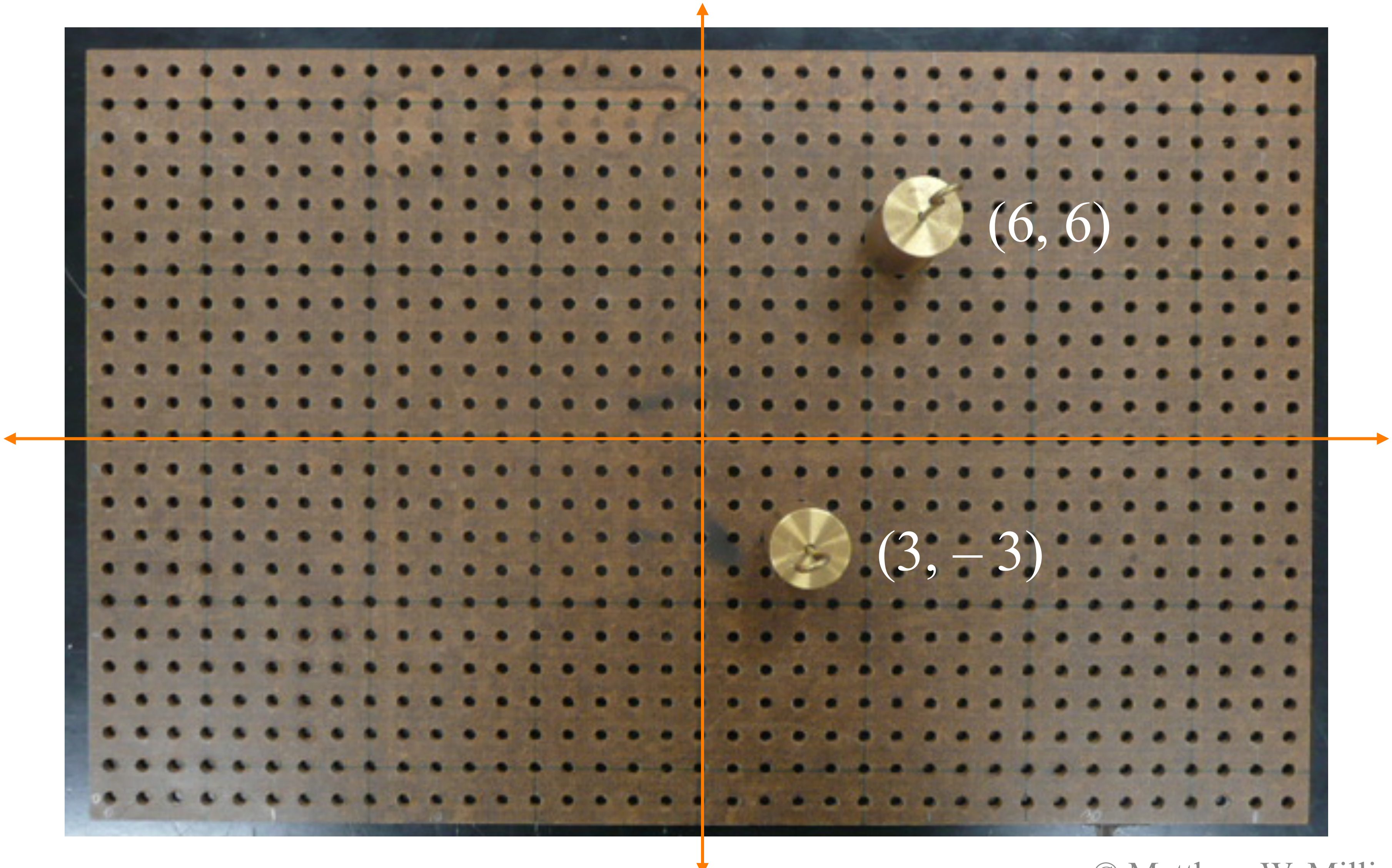


$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + M_3\vec{r}_3}{m_1 + m_2 + M_3}$$

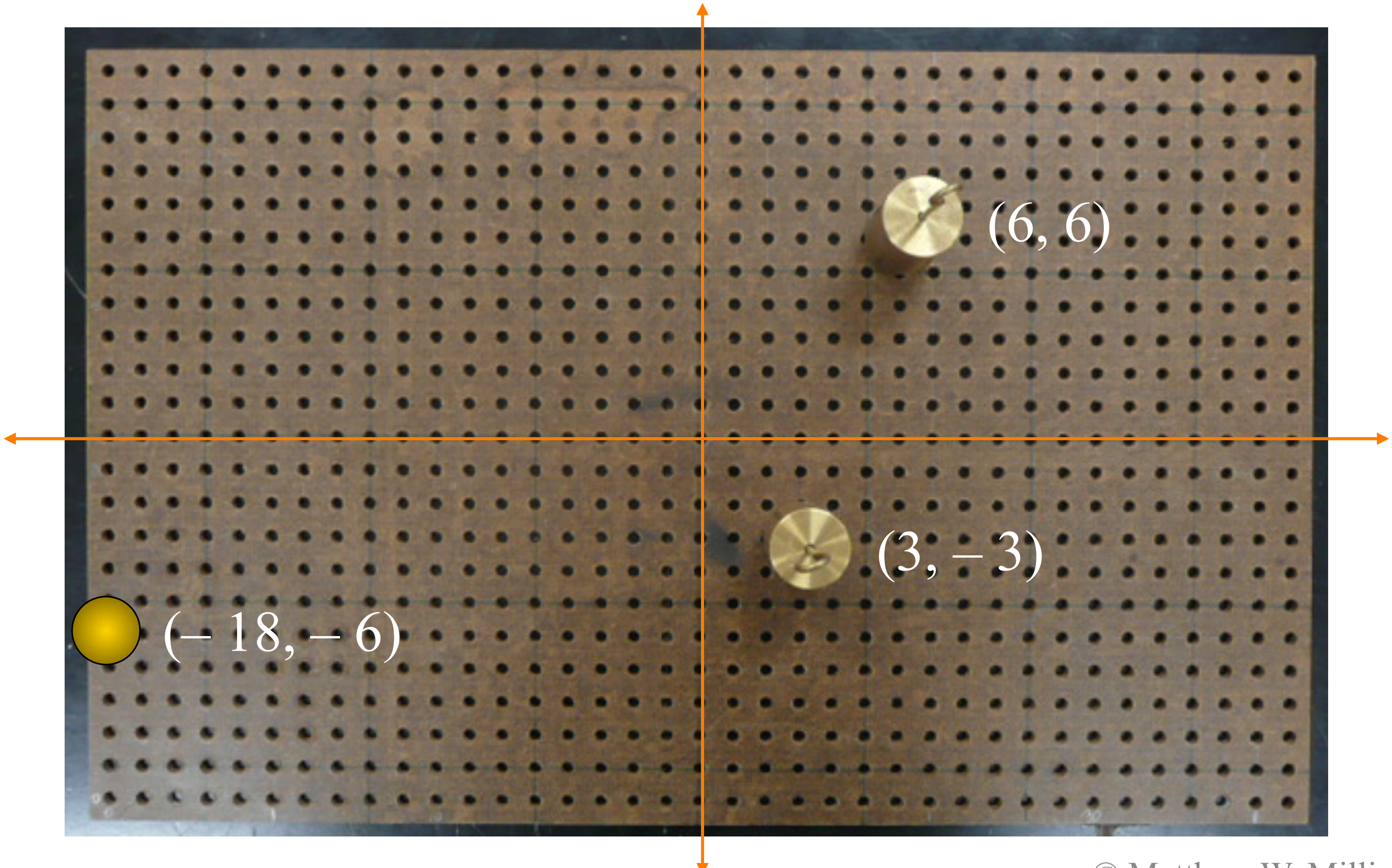


If the *CM* of M_3 is known, then it's position can be used to find the center of mass of the larger set of masses.

At what position would a 100 g mass balance the board and two 200 g masses on the origin?



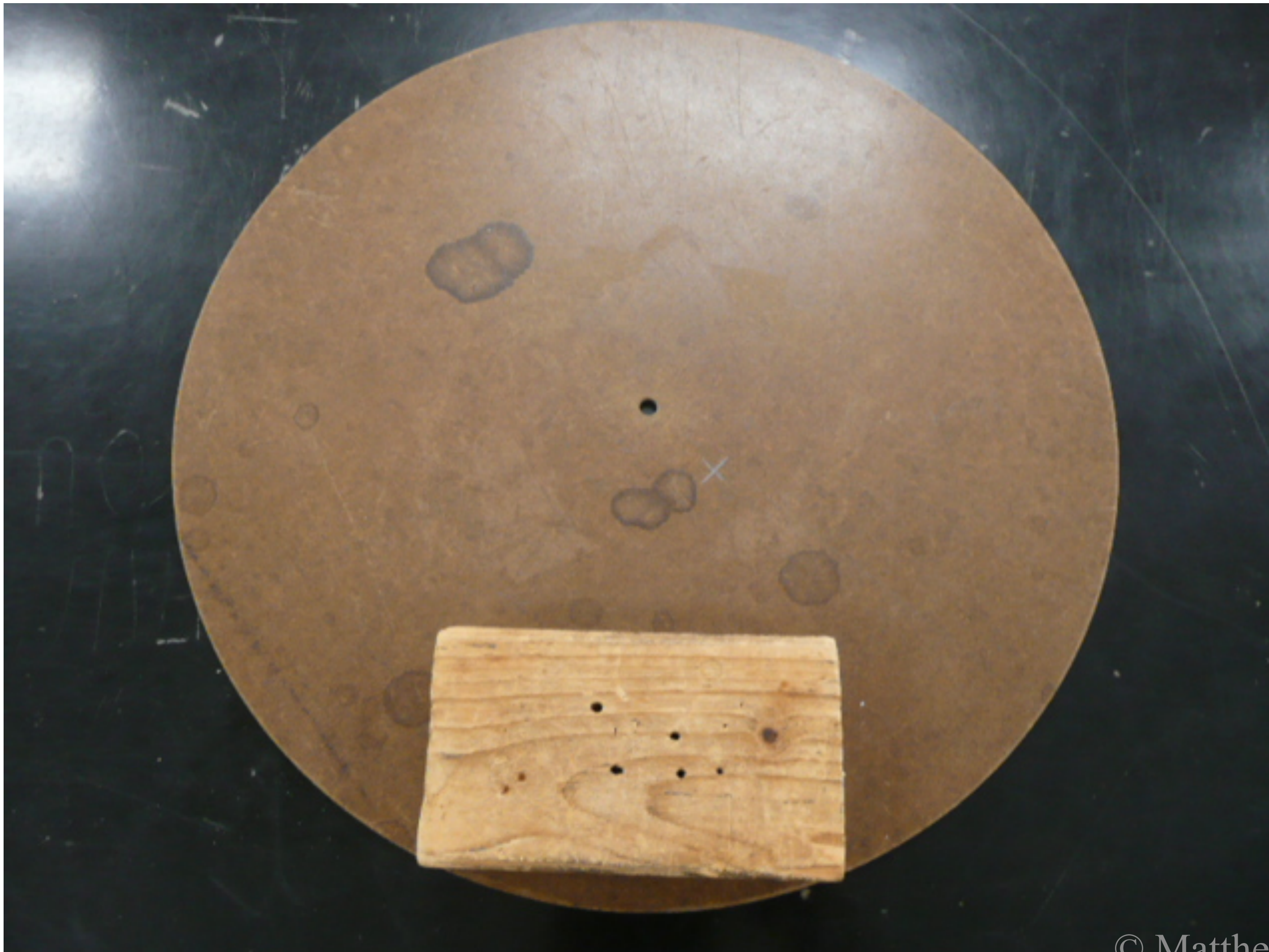
At what position would a 100 g mass balance the board and two 200 g masses on the origin?



Find the location of the center of mass.

disk: $m = 436$ g, $R = 20.2$ cm

block: $m = 160.5$ g, $L = 14.6$ cm, $W = 8.50$ cm.



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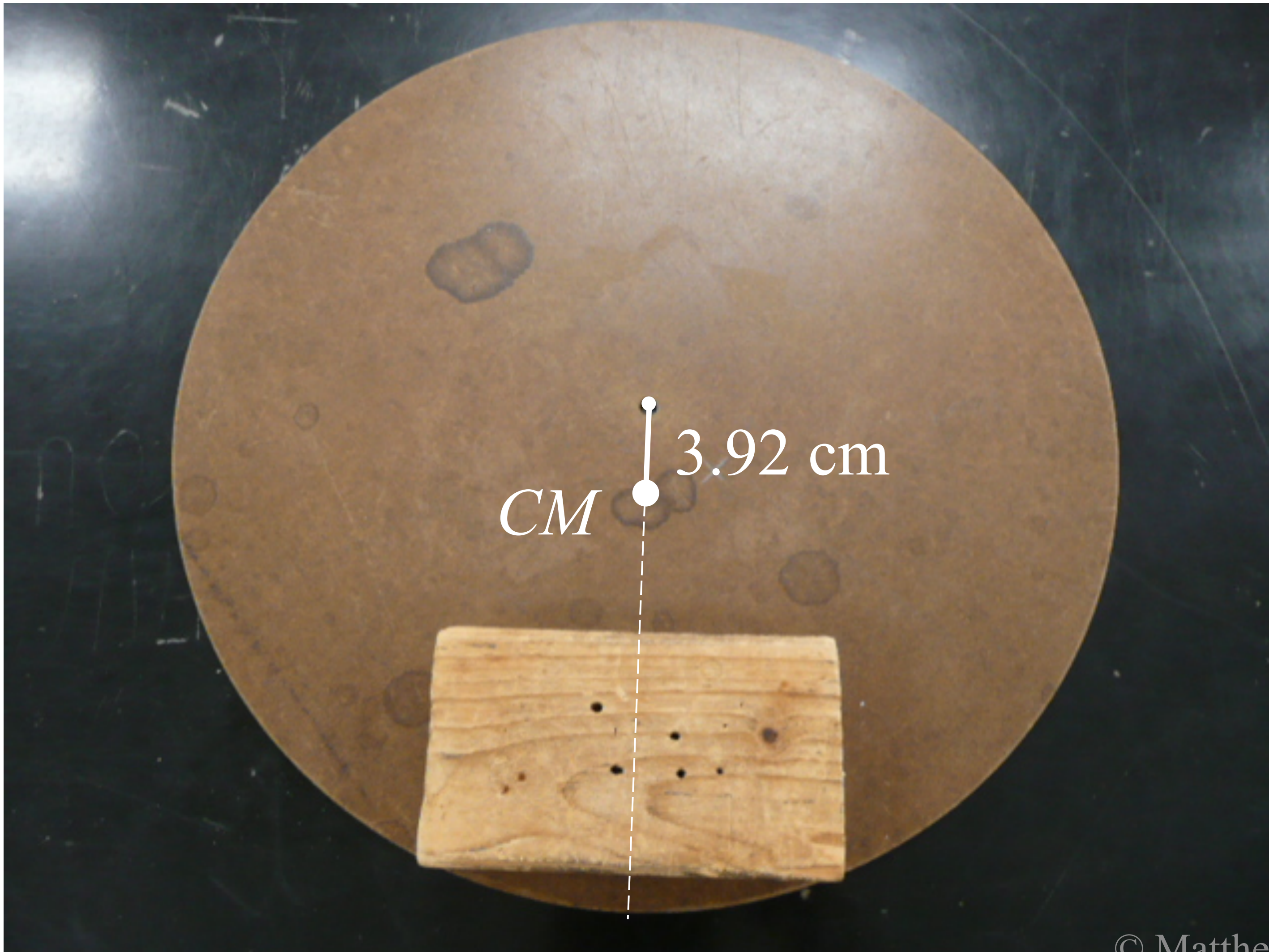
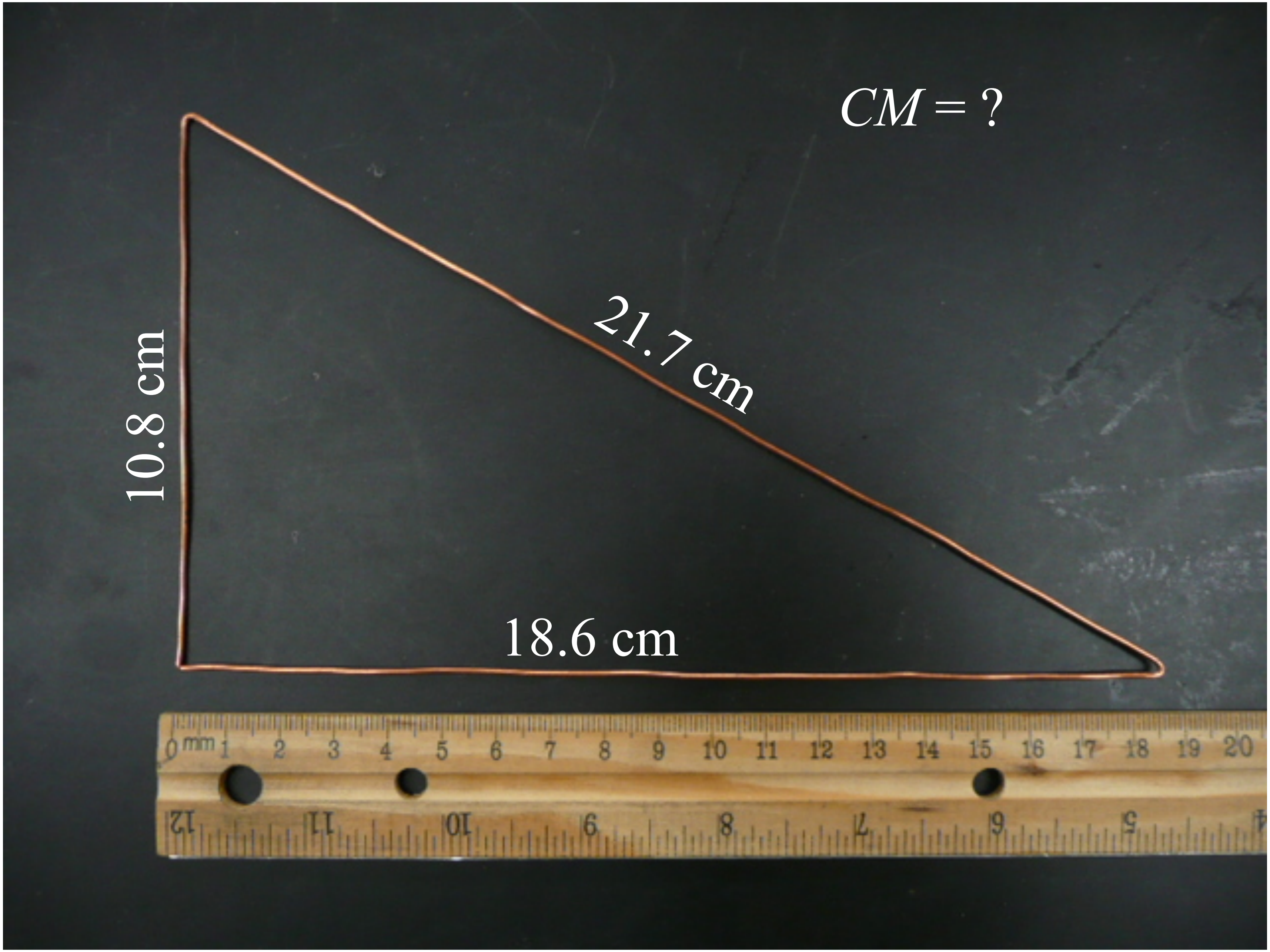
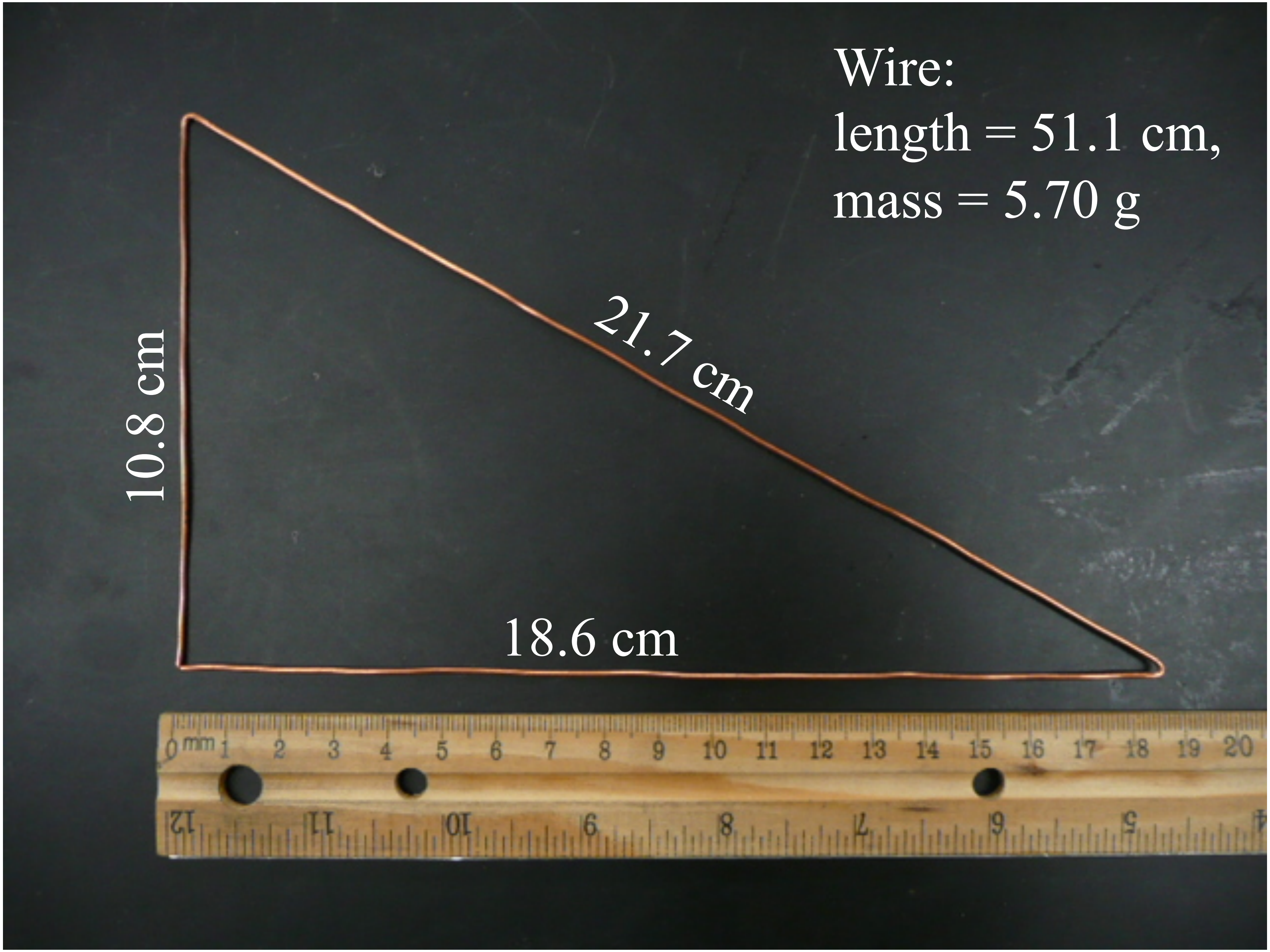
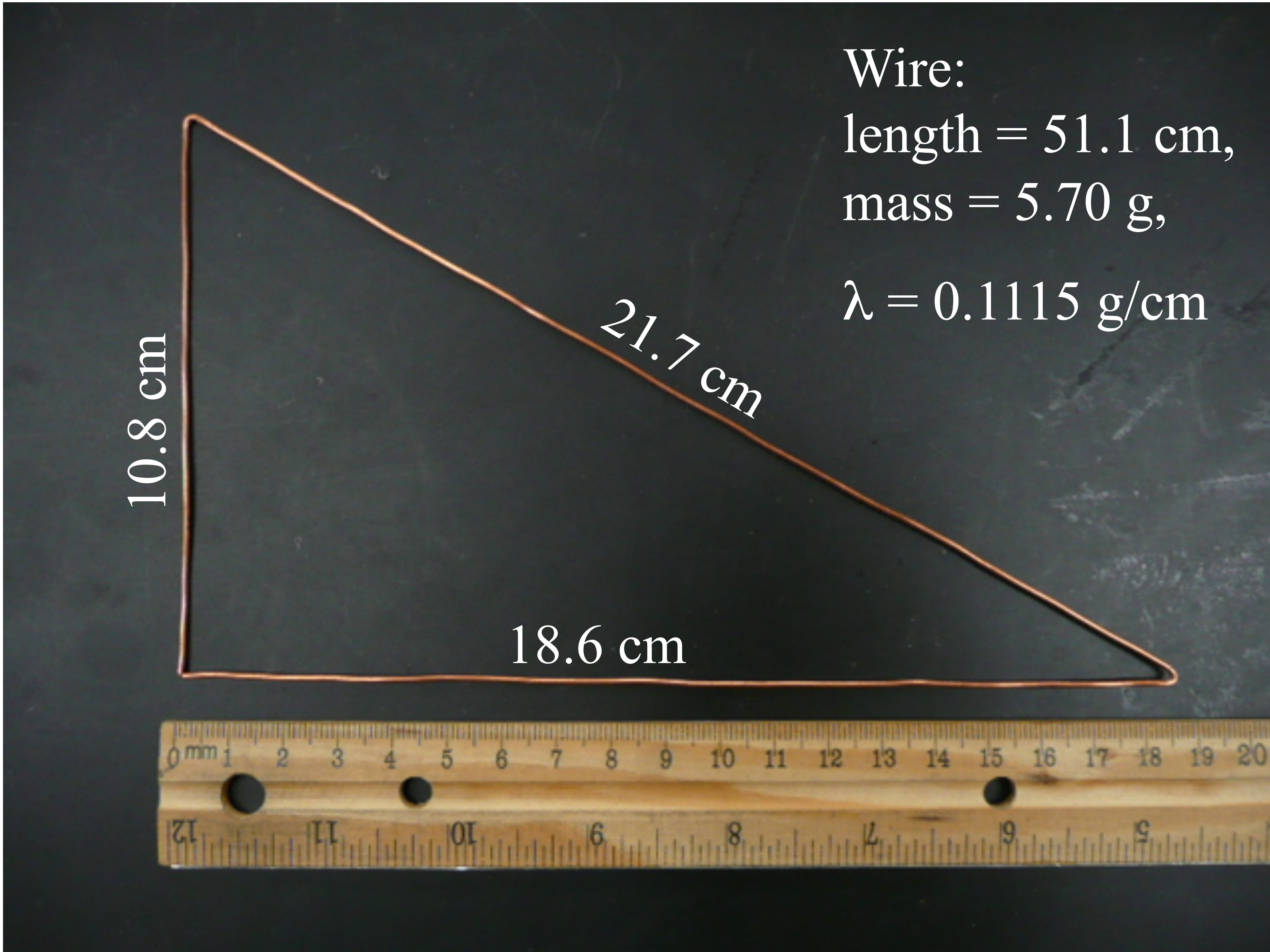
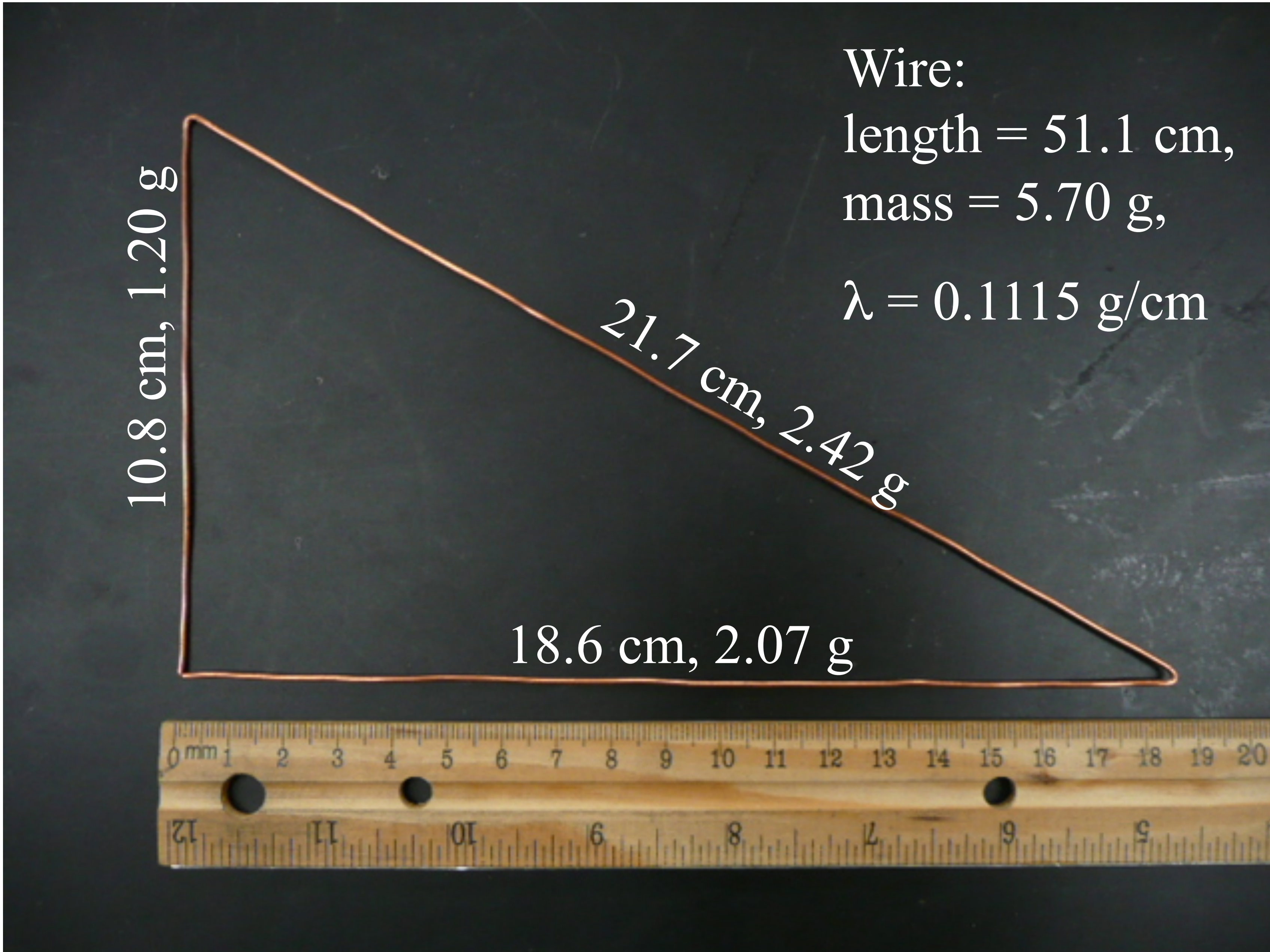


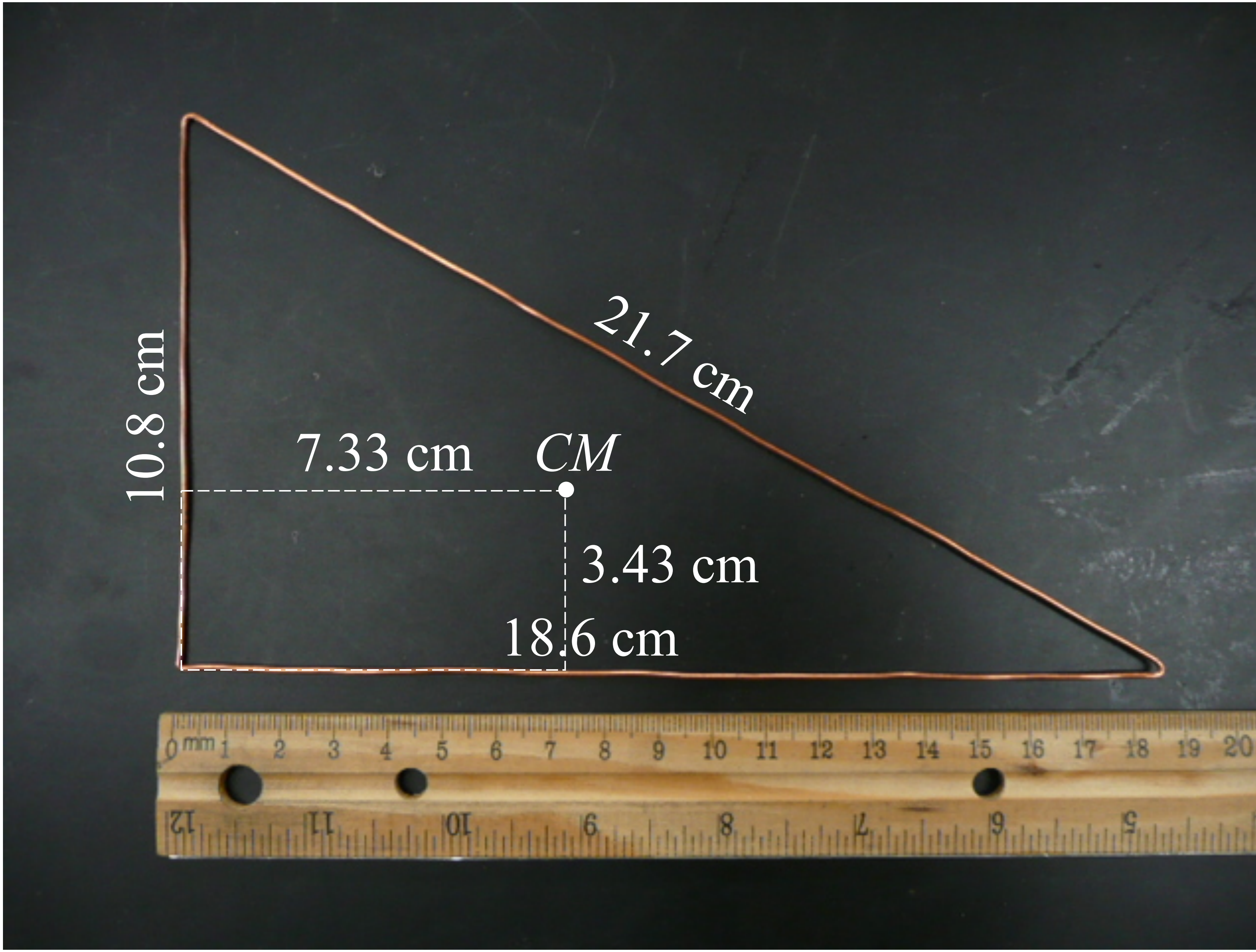


$CM = ?$
10.8 cm
21.7 cm
18.6 cm


Wire:
length = 51.1 cm,
mass = 5.70 g
10.8 cm
21.7 cm
18.6 cm

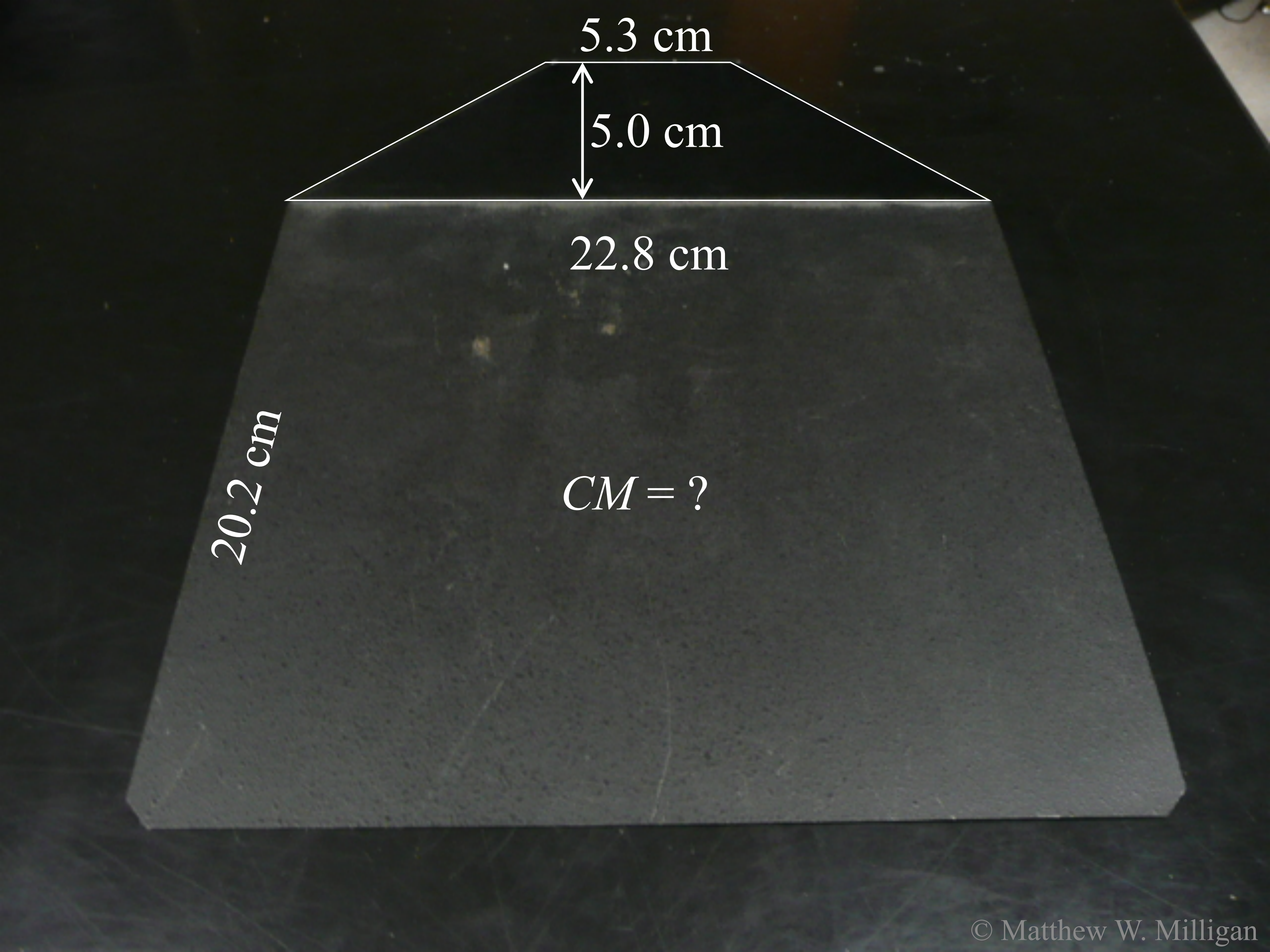

Wire:
length = 51.1 cm,
mass = 5.70 g,
λ = 0.1115 g/cm
21.7 cm
10.8 cm
18.6 cm

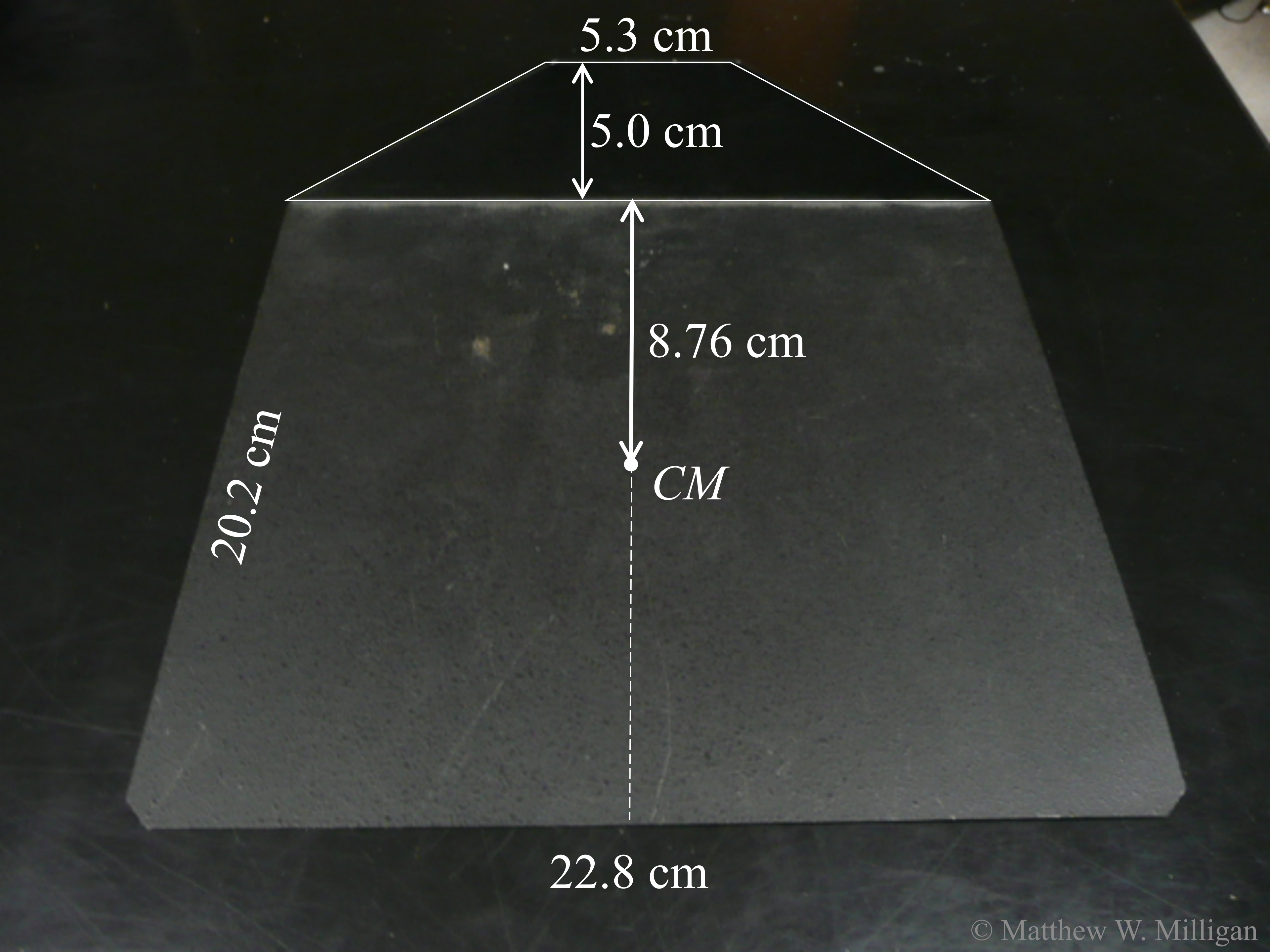

Wire:
length = 51.1 cm,
mass = 5.70 g,
λ = 0.1115 g/cm
10.8 cm, 1.20 g
21.7 cm, 2.42 g
18.6 cm, 2.07 g


21.7 cm
10.8 cm
7.33 cm
CM
3.43 cm
18.6 cm
0 mm 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
12 11 10 9 8 7 6 5 4

Mini-Lab: CM of a Cutout

- Draw and cut out an arbitrary figure, tracing along the gridlines of the index card.
- Pick an origin for an *x*-*y* coordinate system.
- Determine the location of the center of mass of the cutout in this coordinate system.
- Test your result by balancing the cutout.
- Note: this problem can be analyzed in terms of σ = mass/area, assumed to be uniform.
  Index card mass = 1.44 grams.
  Each square is ¼ inch wide.


5.3 cm
5.0 cm
22.8 cm
20.2 cm
CM = ?


5.3 cm
5.0 cm
8.76 cm
CM
20.2 cm
22.8 cm
© Matthew W. Milligan

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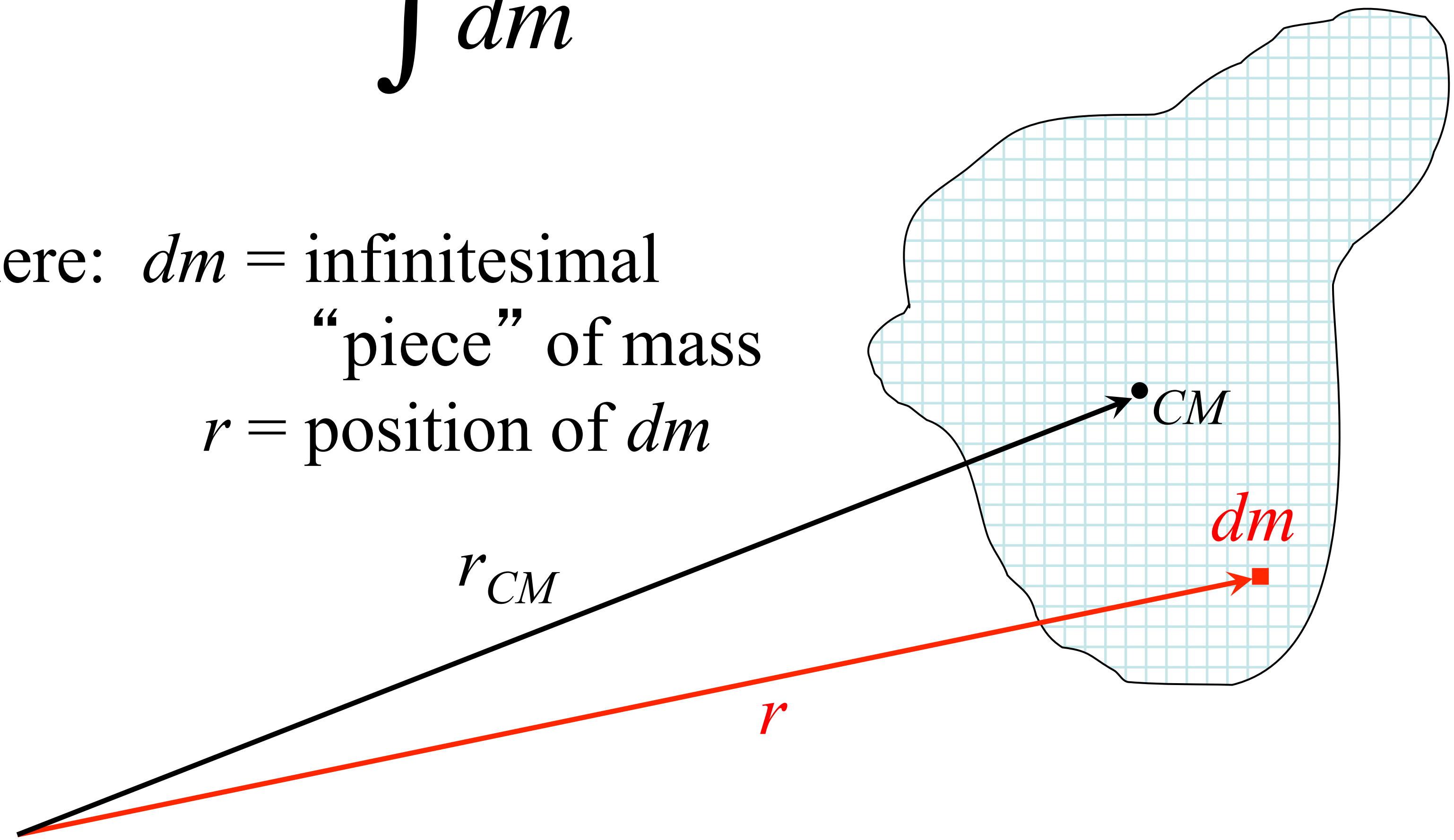
CM' s of "Continuous" Mass

- Often we need to determine the *CM* of a solid object. Such an object is a collection of *discrete* masses (called atoms!)
- However, it is easier to find the *CM* by assuming the mass to be *continuous* and occupying the entire volume of the object.
- It is then possible to find the *CM* by either geometric symmetries or by integration.

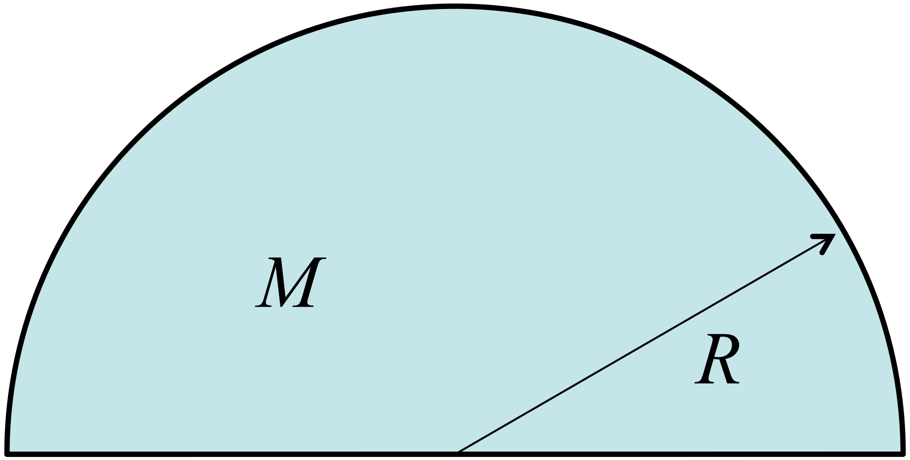
$$\vec{r}_{CM} = \frac{\int \vec{r}\, dm}{\int dm}$$

where: dm = infinitesimal "piece" of mass

r = position of dm



Example finding a center of mass using integration…

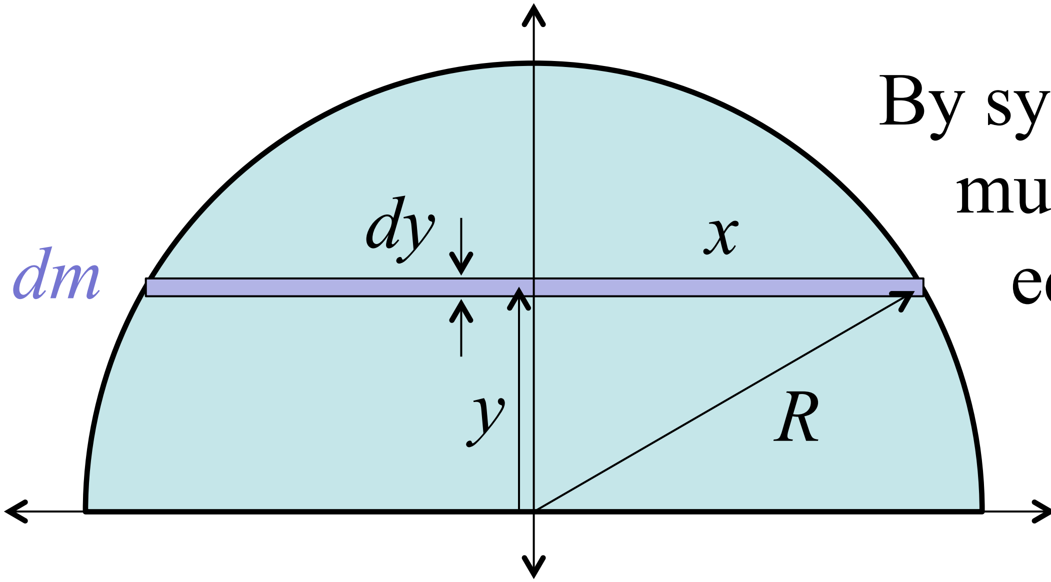


$$\vec{r}_{CM} = \frac{\int \vec{r}\, dm}{\int dm}$$

Find the center of mass of a semicircle of mass M and radius R cut from a thin uniform sheet (like a cutout).

A common technique – define the quantity σ as mass per area. For a uniform material with uniform thickness this is a particular constant. In this case $\sigma = M/(\frac{1}{2}\pi R^2) = 2M/\pi R^2$.

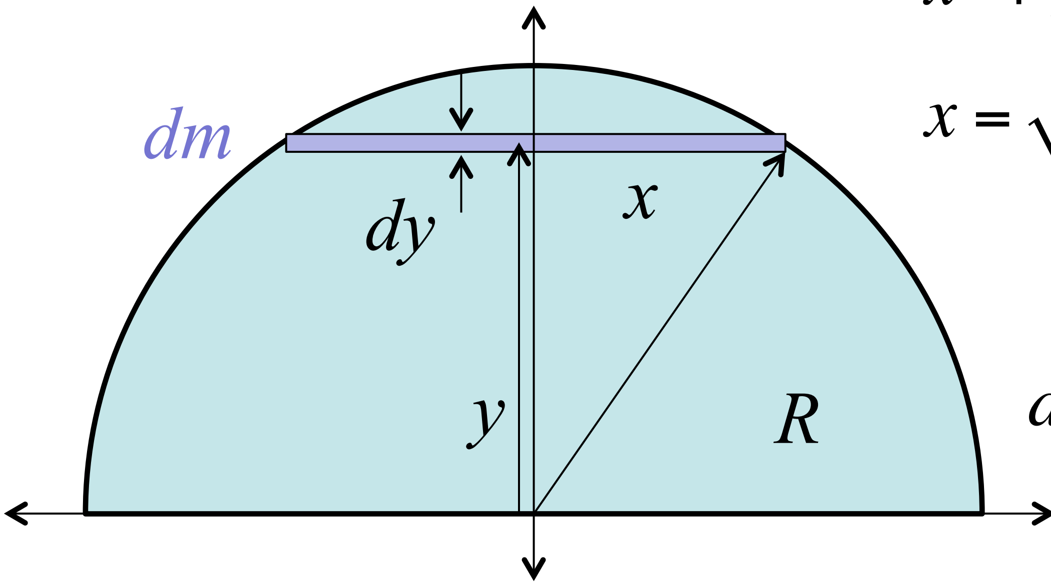
Center of mass semicircular "cutout"



By symmetry the center of mass must lie on the *y*-axis. The equation of the circle is

$$x^2 + y^2 = R^2.$$

What is *dm*? The "*dm*" in the formula represents the mass of an infinitesimal piece of the object. Here it is chosen to be an infinitesimally thin rectangular "slice" of the semicircle. Most of the time it is necessary to put *dm* in terms of other parameters and a chosen variable of integration. The amount of mass in the rectangle is a function of *y* – the greater the value of *y*, the shorter the rectangle and the less its mass.



$$x^2 + y^2 = R^2$$

$$x = \sqrt{R^2 - y^2}$$

$$dA = (2x)dy$$

$$dA = 2\sqrt{R^2 - y^2}\,dy$$

$$\sigma = \frac{dm}{dA}$$

$$dm = \sigma dA$$

$$dm = \frac{2M}{\pi R^2} 2x dy$$

$$dm = \frac{4M}{\pi R^2}\sqrt{R^2 - y^2}\,dy$$

(dm in terms of y)

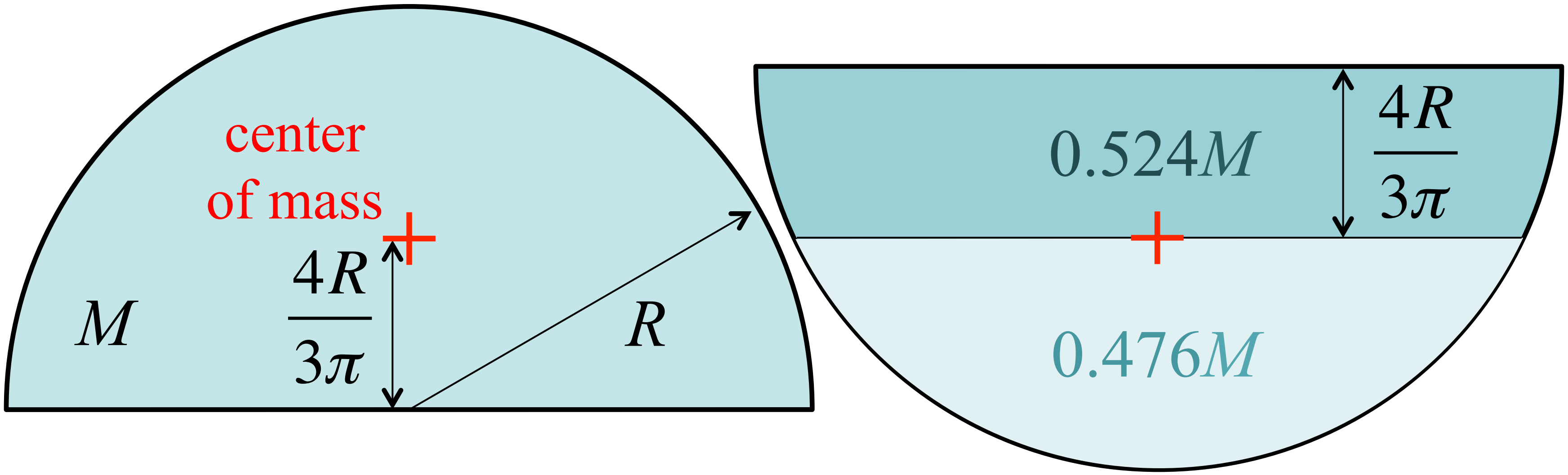
$$y_{CM} = \frac{\int y\,dm}{\int dm} = \frac{\int y\,dm}{M}$$

$$y_{CM} = \frac{\int_0^R y \frac{4M}{\pi R^2}\sqrt{R^2 - y^2}\,dy}{M}$$

$$y_{CM} = \frac{4}{\pi R^2}\int_0^R y\sqrt{R^2 - y^2}\,dy$$

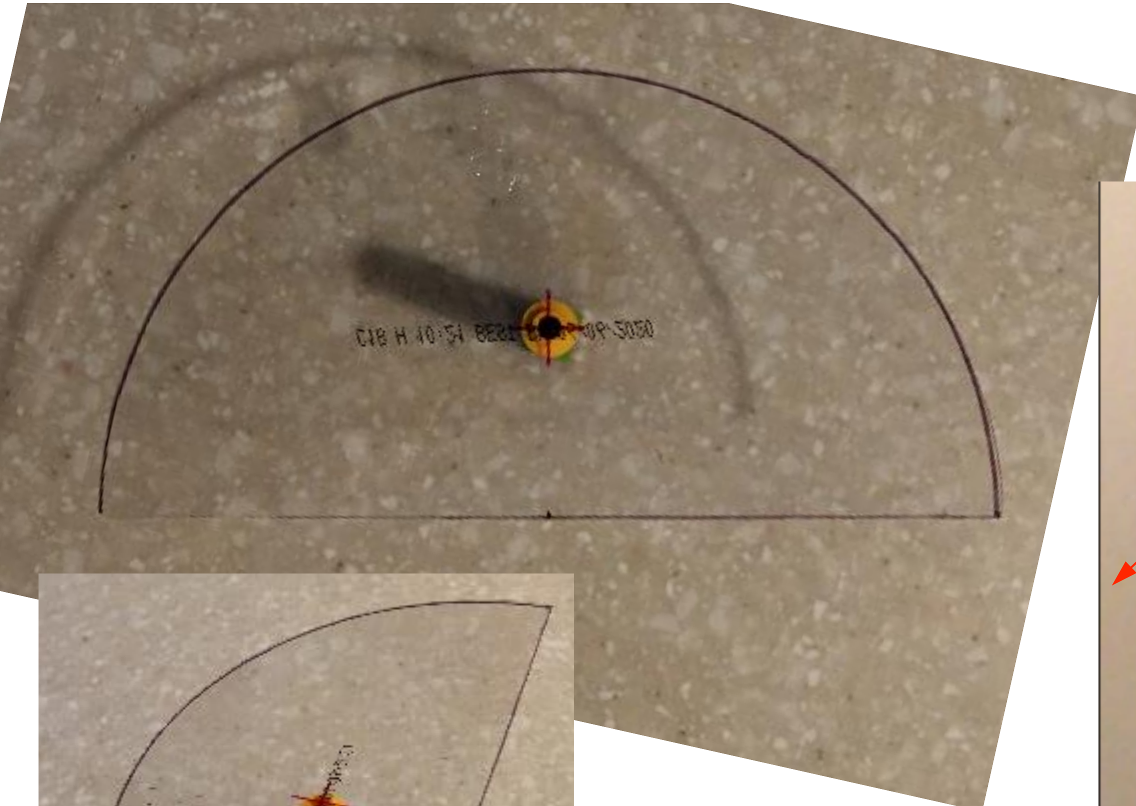
$$y_{CM} = \frac{4}{\pi R^2}\frac{\left(R^2\right)^{\frac{3}{2}}}{3}$$

$$y_{CM} = \frac{4R}{3\pi}$$

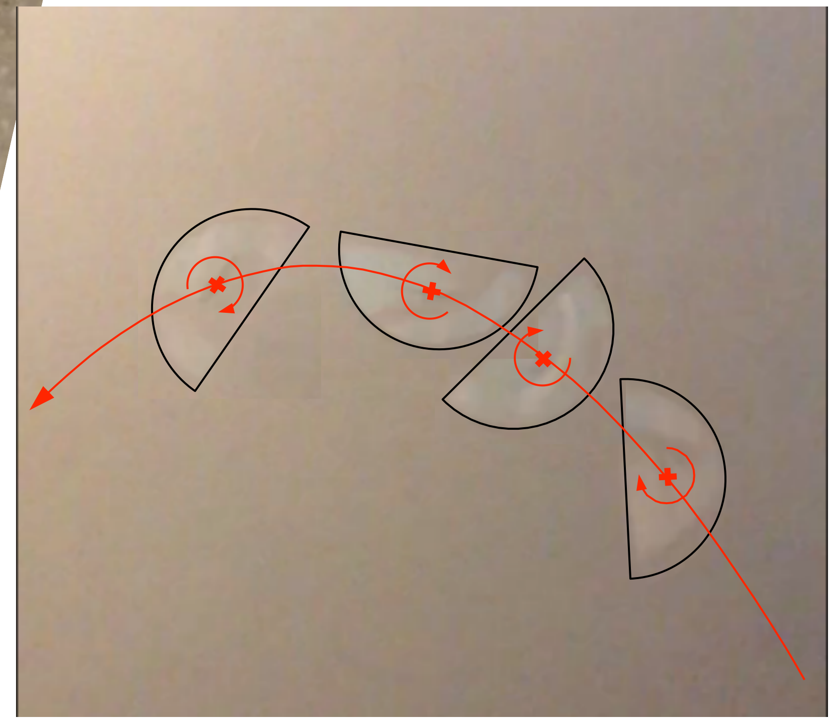


It is intuitively obvious that the center of mass is located on a *vertical* line of symmetry that divides the semicircle into two equal quarter circles of *equal* mass. But, don't get the wrong idea! A *horizontal* line passing through the center of mass does *not* divide the semicircle into two equal masses! A line through the center of mass and parallel to the diameter divides the mass into *unequal* parts 52.4% and 47.6% of the total.

Actual photos of a real semicircle cutout from a plastic lid:



Center of mass is marked. The cutout balances on this point.



Spinning through the air, the center of mass follows a parabolic path!