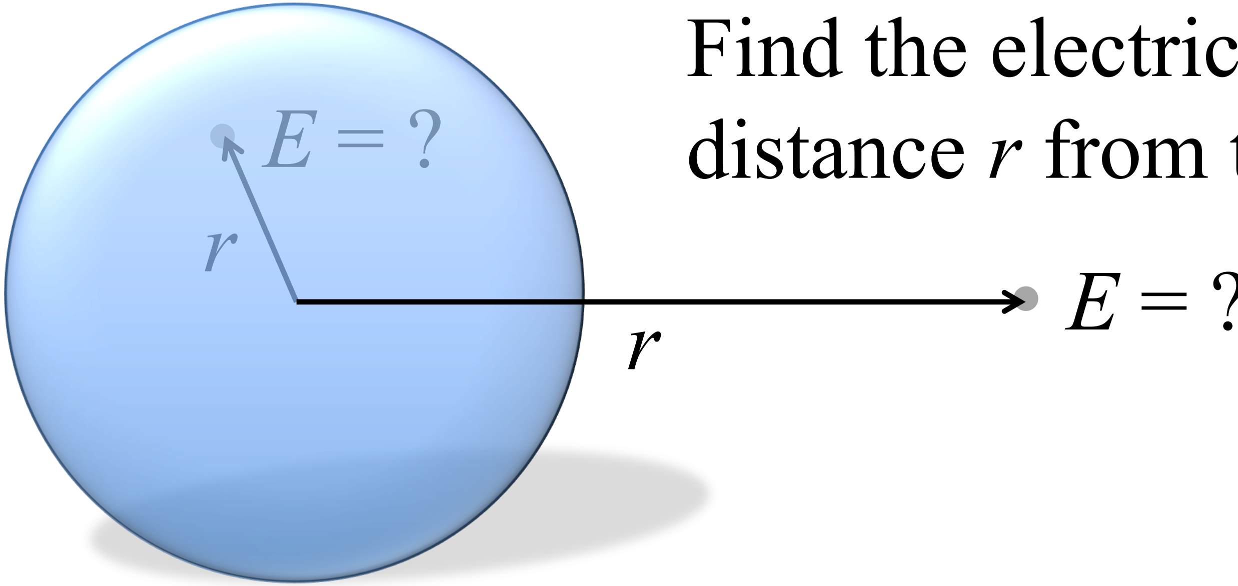
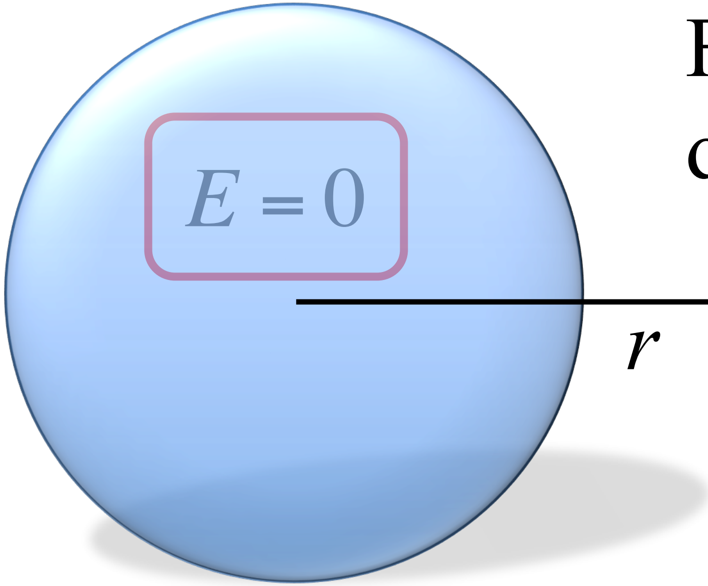
Gauss's Law vs. Coulomb's Law

An Appreciation

Find the electric field E at distance r from the center…



Spherical surface of radius R and uniform charge Q.



Find the electric field E at distance r from the center…

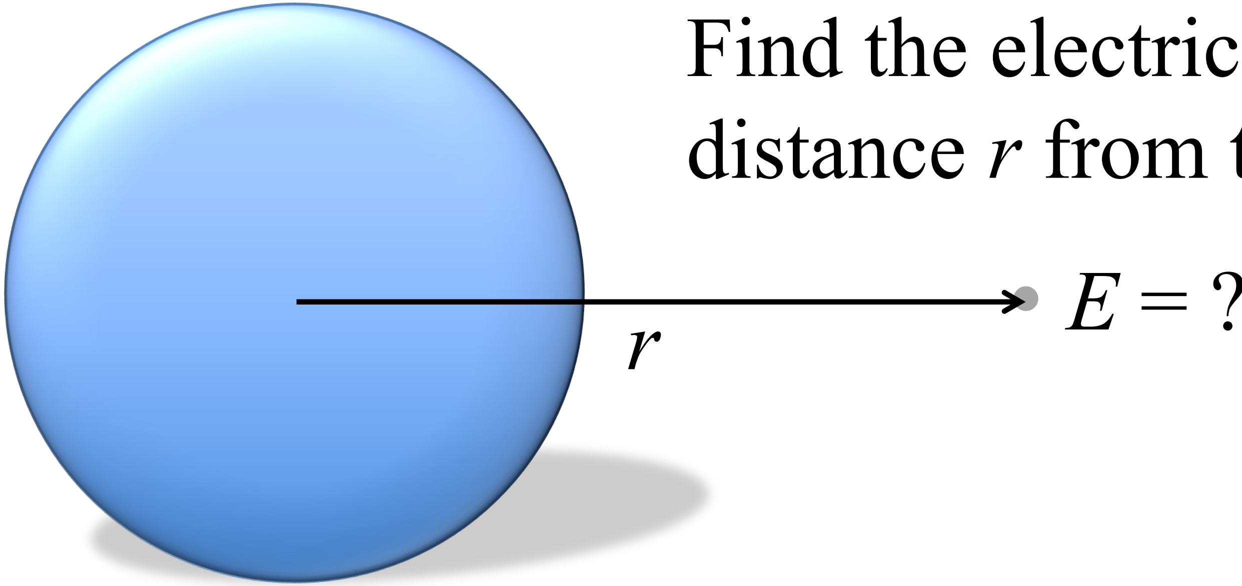
$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

Spherical surface of radius R and uniform charge Q.

It is ***easy*** to derive these results using Gauss's Law: The field anywhere inside is zero and anywhere outside is equivalent to that of a point charge.

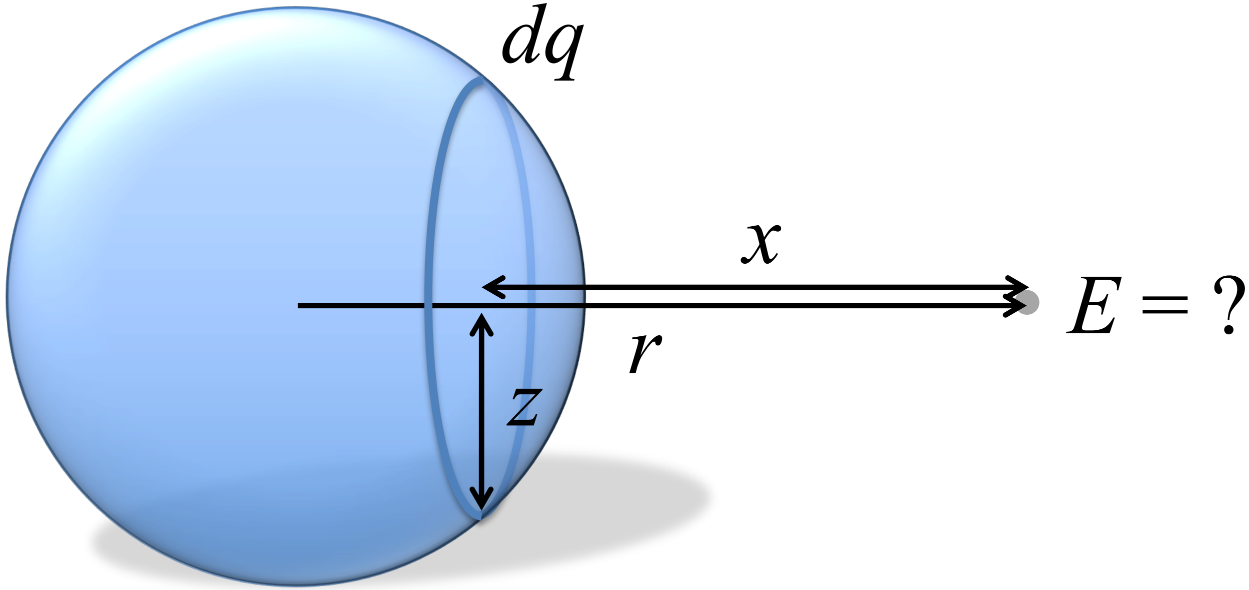
BUT, what if you did *not* use Gauss's Law?! …

Find the electric field E at distance r from the center…

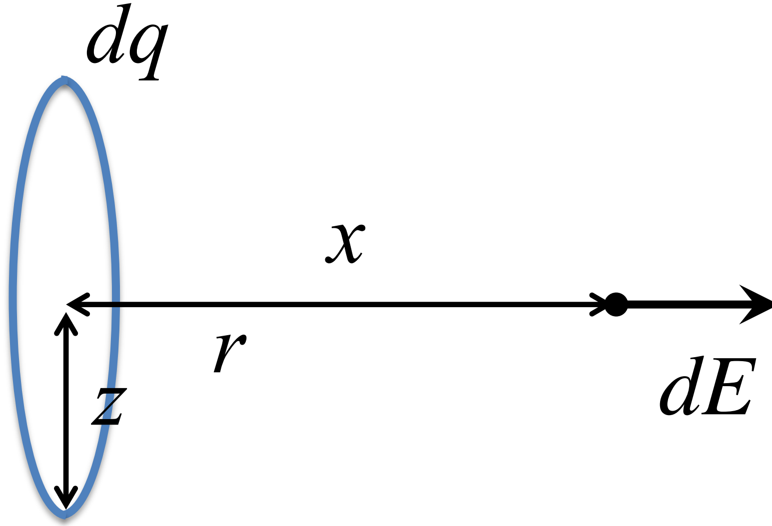


Spherical surface of radius R and uniform charge Q.

Charge per area: $\sigma = \dfrac{Q}{4\pi R^2}$

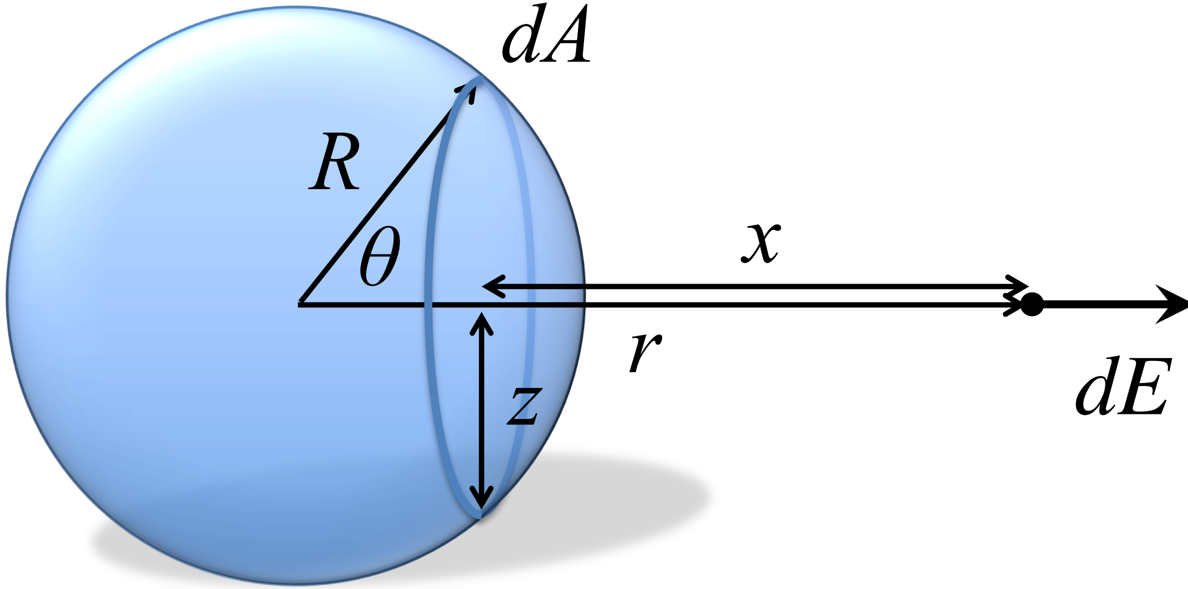


Slice sphere into rings of radius z and charge dq, each at distance x from the point in question…



$$dE = \frac{x\,dq}{4\pi\varepsilon_0\left(z^2 + x^2\right)^{\frac{3}{2}}}$$

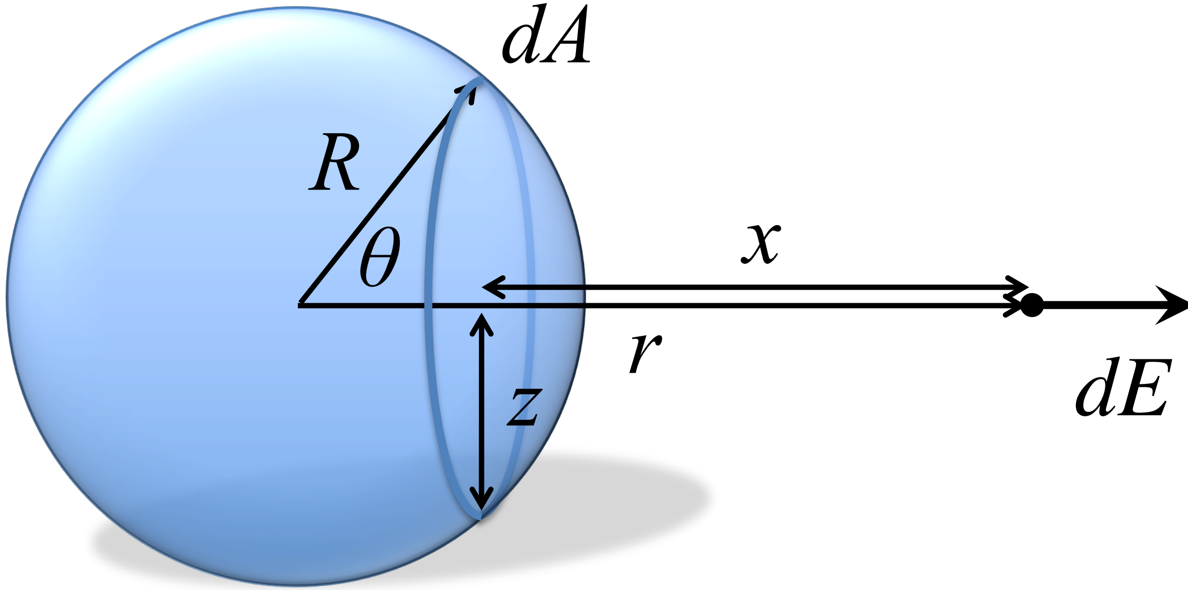
Field contribution, *dE*, of each ring (previously derived using Coulomb's Law and inverse square property of point charges)



$$dE = \frac{x\,dq}{4\pi\varepsilon_0\left(z^2 + x^2\right)^{\frac{3}{2}}}$$

Area of each ring is given by: $dA = 2\pi zRd\theta$ and therefore dq is: $dq = \sigma 2\pi zRd\theta$

$$dE = \frac{x\sigma 2\pi zRd\theta}{4\pi\varepsilon_0\left(z^2 + x^2\right)^{\frac{3}{2}}}$$



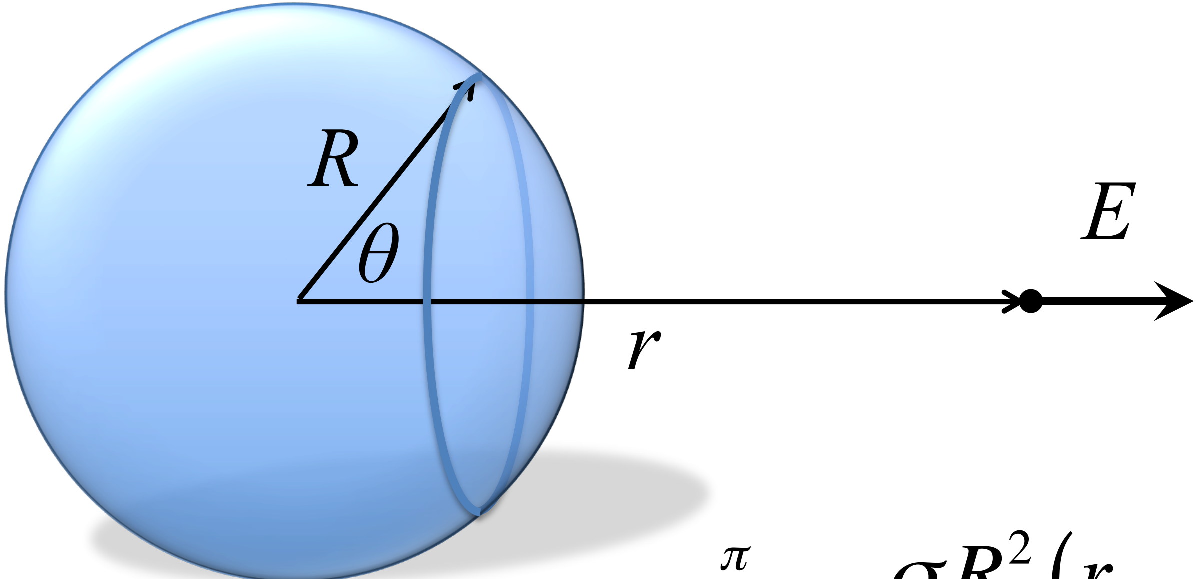
$$dE = \frac{x\sigma 2\pi z R d\theta}{4\pi\varepsilon_0\left(z^2 + x^2\right)^{\frac{3}{2}}}$$

Geometry:

$$z = R\sin\theta$$

$$x = r - R\cos\theta$$

$$dE = \frac{\left(r - R\cos\theta\right)\sigma 2\pi R^2 \sin\theta d\theta}{4\pi\varepsilon_0\left(\left(R\sin\theta\right)^2 + \left(r - R\cos\theta\right)^2\right)^{\frac{3}{2}}}$$



$$E = \int_0^{\pi} \frac{\sigma R^2 \left(r - R\cos\theta\right)\sin\theta \, d\theta}{2\varepsilon_0 \left(R^2 \sin^2\theta + \left(r - R\cos\theta\right)^2\right)^{\frac{3}{2}}}$$

This should do it! All that remains is to evaluate…

$$E=\int_0^\pi \frac{\sigma R^2\left(r-R\cos\theta\right)\sin\theta d\theta}{2\varepsilon_0\left(R^2\sin^2\theta+\left(r-R\cos\theta\right)^2\right)^{\frac{3}{2}}}$$

$$E=\int_0^\pi \frac{\sigma R^2\left(r-R\cos\theta\right)\sin\theta d\theta}{2\varepsilon_0\left(R^2\sin^2\theta+r^2-2Rr\cos\theta+R^2\cos^2\theta\right)^{\frac{3}{2}}}$$

$$E=\int_0^\pi \frac{\sigma R^2\left(r-R\cos\theta\right)\sin\theta d\theta}{2\varepsilon_0\left(R^2\left(\sin^2\theta+\cos^2\theta\right)+r^2-2Rr\cos\theta\right)^{\frac{3}{2}}}$$

$$E=\int_0^{\pi}\frac{\sigma R^2\left(r-R\cos\theta\right)\sin\theta d\theta}{2\varepsilon_0\left(R^2+r^2-2Rr\cos\theta\right)^{\frac{3}{2}}}$$

$$u=R^2+r^2-2Rr\cos\theta$$

$$\cos\theta=\frac{R^2+r^2-u}{2Rr}$$

$$E=\frac{\sigma R^2}{2\varepsilon_0}\int_0^{\pi}\frac{u^{-\frac{3}{2}}}{2R}-\frac{\left(R^2+r^2\right)u^{-\frac{3}{2}}}{4Rr^2}+\frac{u^{-\frac{1}{2}}}{4Rr^2}du$$

$$\frac{du}{d\theta}=2Rr\sin\theta$$

$$E=\frac{\sigma R^2}{2\varepsilon_0}\int_0^{\theta=\pi}\frac{u^{-\frac{3}{2}}}{2R}-\frac{\left(R^2+r^2\right)u^{-\frac{3}{2}}}{4Rr^2}+\frac{u^{-\frac{1}{2}}}{4Rr^2}du$$

$$\frac{du}{2Rr}=\sin\theta\,d\theta$$

$$E=\frac{\sigma R^2}{2\varepsilon_0}\left(-\frac{1}{R\sqrt{u}}+\frac{R^2+r^2}{2Rr^2\sqrt{u}}+\frac{\sqrt{u}}{2Rr^2}\right)_{\theta=0}^{\theta=\pi}$$

$$E = \frac{\sigma R^2}{2\varepsilon_0}\left(-\frac{1}{R\sqrt{u}} + \frac{R^2 + r^2}{2Rr^2\sqrt{u}} + \frac{\sqrt{u}}{2Rr^2}\right)_{\theta=0}^{\theta=\pi}$$

$$E = \frac{\sigma R^2}{2\varepsilon_0}\left(\frac{-2r^2 + R^2 + r^2 + u}{2Rr^2\sqrt{u}}\right)_{\theta=0}^{\theta=\pi} \qquad u = R^2 + r^2 - 2Rr\cos\theta$$

$$E = \frac{\sigma R^2}{2\varepsilon_0}\left(\frac{-2r^2 + R^2 + r^2 + R^2 + r^2 - 2Rr\cos\theta}{2Rr^2\sqrt{u}}\right)_{\theta=0}^{\theta=\pi}$$

$$E = \frac{\sigma R^2}{2\varepsilon_0}\left(\frac{2R^2 - 2Rr\cos\theta}{2Rr^2\sqrt{u}}\right)_{\theta=0}^{\theta=\pi}$$

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$$E = \frac{\sigma R^2}{2\varepsilon_0}\left(\frac{R - r\cos\theta}{r^2\sqrt{u}}\right)_{\theta=0}^{\theta=\pi}$$

$$E = \frac{\sigma R^2}{2\varepsilon_0}\left(\frac{R - r\cos\theta}{r^2\sqrt{R^2 + r^2 - 2Rr\cos\theta}}\right)_{\theta=0}^{\theta=\pi}$$

$$E = \frac{\sigma R^2}{2\varepsilon_0}\left(\frac{R - r(-1)}{r^2\sqrt{R^2 + r^2 - 2Rr(-1)}} - \frac{R - r(1)}{r^2\sqrt{R^2 + r^2 - 2Rr(1)}}\right)$$

$$E = \frac{\sigma R^2}{2\varepsilon_0}\left(\frac{R+r}{r^2\sqrt{r^2+2Rr+R^2}} - \frac{R-r}{r^2\sqrt{r^2-2Rr+R^2}}\right)$$

$$E = \frac{\sigma R^2}{2\varepsilon_0}\left(\frac{R+r}{r^2\sqrt{(r+R)^2}} + \frac{r-R}{r^2\sqrt{(r-R)^2}}\right)$$

If $r > R$

$$E = \frac{\sigma R^2}{2\varepsilon_0}\left(\frac{1}{r^2} + \frac{1}{r^2}\right)$$

$$E = \frac{2\sigma R^2}{2\varepsilon_0 r^2}$$

$$E = \frac{\frac{Q}{4\pi R^2}R^2}{\varepsilon_0 r^2}$$

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

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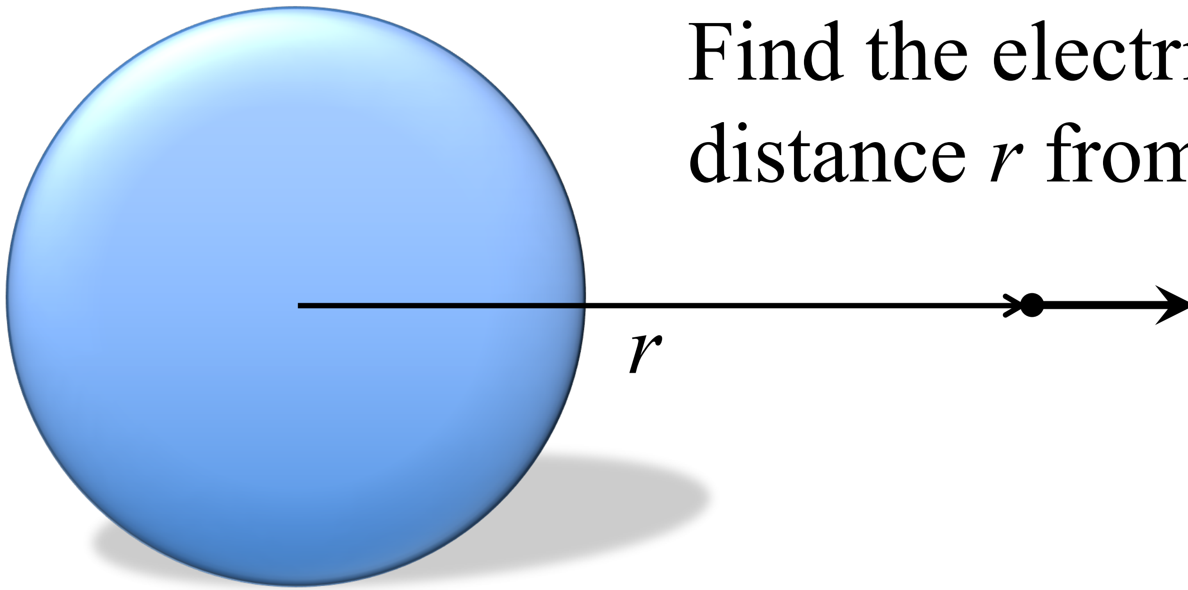


$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

Spherical surface of radius R and uniform charge Q.

$$E = \frac{\sigma R^2}{2\varepsilon_0}\left(\frac{R+r}{r^2\sqrt{r^2+2Rr+R^2}} - \frac{R-r}{r^2\sqrt{r^2-2Rr+R^2}}\right)$$

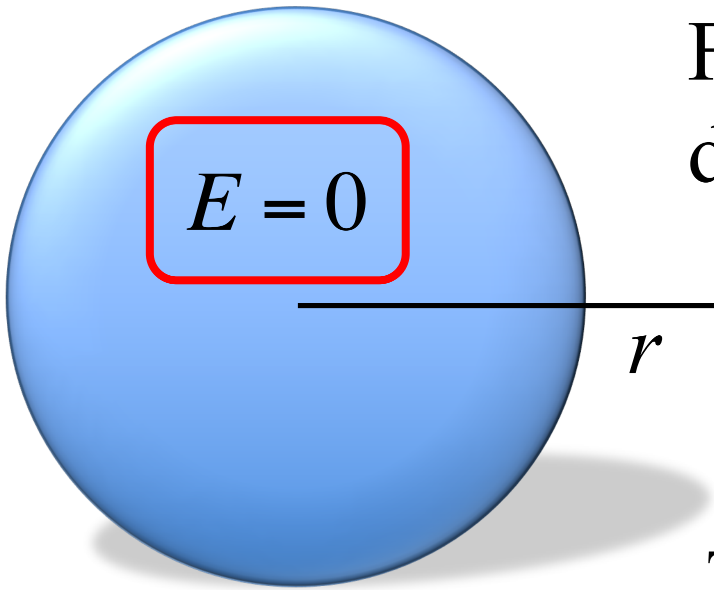
$$E = \frac{\sigma R^2}{2\varepsilon_0}\left(\frac{R+r}{r^2\sqrt{(r+R)^2}} + \frac{r-R}{r^2\sqrt{(r-R)^2}}\right)$$

If $r < R$

$$E = \frac{\sigma R^2}{2\varepsilon_0}\left(\frac{1}{r^2} + \frac{-1}{r^2}\right)$$

$$E = 0$$

Find the electric field E at distance r from the center…



$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

Spherical surface of radius R and uniform charge Q.

The exact same result for both outside and inside can be derived based on Coulomb's Law – but just think how much easier it is using Gauss's Law! Thank you Mr. Gauss, I appreciate you!!!!